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DRAWING AN ANALOGY BETWEEN THE DIRAC–MAXWELL–EINSTEIN THEORY AND A FIELD MODEL FOR SEMIONS

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We show that there is much in common between the Dirac - Maxwell - Einstein (DME) theory and a semion model in 2-dimensional Grassmann space which allows one to treat a supersymmetric semion model in $D = 2 + 1$ space-time [1] as an effective theory resulting from the 2-dimensional model.

Semions (that is particles with spin 1/4 and 3/4 in $D = 2 + 1$), being particular representatives of the anyons [2], reveal interesting field-theoretical [3,4] and interaction [5] properties and, as one may hope, are relevant to high-T_c superconductivity and other problems of strongly correlated quantum electron systems in 2-dimensional space [6].

In the present letter by drawing an analogy with the DME theory we construct a field-theoretical semion model in 2-dimensional Grassmann space which is effectively equivalent to a $D = 2 + 1$ supersymmetric semion model [1] in the momentum representation, with space-time dynamical characteristics of the latter arising as a manifestation of a gauge and a gravitational field of the 2-dimensional model.

To make the idea most transparent we consider the grounds of both the DME theory and the semion model in parallel. The following notation and convention are used: $(l, m, n, ...)$ stand for vector indices, in particular in $D = 2 + 1$ space-time; $(\alpha, \beta, \gamma, ... = 1, 2)$ are $SL(2, R)$ spinor indices, which are raised and lowered by the unit skew-symmetric matrices $\varepsilon^{\alpha\beta} = \varepsilon_{\alpha\beta}$; and the signature of the Minkowski space-time metric g_{mn} is chosen to be $(+, -..., -)$.

Lorentz group representation. To describe fermions one introduces D-dimensional vector matrices γ^m which generate the Clifford algebra

$$\{\gamma^m, \gamma^n\} \equiv \gamma^m\gamma^n + \gamma^n\gamma^m = 2g^{mn}. \tag{1}$$

The generators of the spinor representation of the Lorentz group are realized as the commutators of γ^m:

$$S^{mn} = \frac{i}{4}[\gamma^m, \gamma^n]. \tag{2}$$

To describe semions one introduces spinor operators L^α which generate the Heisenberg algebra [7,4,8]

$$[L^\alpha, L^\beta] \equiv L^\alpha L^\beta - L^\beta L^\alpha = i\varepsilon^{\alpha\beta} \tag{1.a}$$

The generators of $SL(2, R)$ (being isomorphic to the $D = 2 + 1$ Lorentz group) are realized, for the case of semions, as the anti-commutators of L^α :

$$S^{\alpha\beta} = \frac{1}{2}\{L^\alpha, L^\beta\} \tag{2.a}$$

An irreducible representation of the Heisenberg - Weyl group generated by the algebra (1.a) splits into irreducible $SL(2,R)$ representations with the weights 1/4 and 3/4 describing semions (see Ref. [4,1] for the details).

In view of the anti-commutator in (2.a), we consider L^α to be odd and the relative statitstics of the $SL(2,R)$ representations to be fermionic.

One can see that γ^m and L^α are the antipodes in the sense that where the commutator arises for the Dirac matrices the anti-commutator arises for L^α and vice versa. Below we use this interchange in the commuting and anti-commuting properties for constructing the semion model in analogy with the DME theory.

Free field equations. Fermions are known to propagate in the space-time parametrized by bosonic vector coordinates x^m and to be described by the Dirac equation

$$\gamma^m \partial_m \psi(x) = 0, \tag{3}$$

where $\psi_\alpha(x)$ is a fermion wave function. Here we restrict the consideration to the massless fermions. See below a comment on the analogy between the massive Dirac theory and the semion model.

Semions are assumed to propagate in a Grassmann spinor space parametrized by odd Majorana coordinates θ^α and to be described by the equation

$$L^\alpha \frac{\partial}{\partial\theta^\alpha}\Phi(\theta) = 0, \tag{3.a}$$

where $\Phi(\theta) = A + i\theta^\alpha\psi_\alpha + i\theta^\alpha\theta_\alpha C$ is a semion wave function transforming as the Heisenberg - Weyl group representation.

Interaction with an abelian gauge field. To consider the interaction of fermions with an (external) Maxwell field $A_m(x)$ we write down the Dirac equation in the form

$$\gamma^m D_m\psi \equiv \gamma^m(\partial_m + iA_m(x))\psi(x) = 0 \tag{4}$$

covariant under the abelian gauge transformations

$$\psi(x) \to \psi(x)\exp i\varphi(x),$$

$$A_m(x) \to A_m(x) - \partial_m\varphi(x). \tag{5}$$

In the same way we introduce the interaction of semions with an odd abelian gauge field $A_\alpha(\theta) = a_\alpha + \theta^\beta p_{\beta\alpha} + i\theta^\beta\theta_\beta c_\alpha$ by generalizing Eq. (3.a) to the form

$$L^\alpha D_\alpha\Phi(\theta) \equiv L^\alpha(\partial_\alpha + A_\alpha(\theta))\Phi(\theta) = 0 \tag{4.a}$$

covariant under abelian gauge transformations

$$\Phi(\theta) \to \Phi(\theta)\exp i\varphi(\theta),$$

$$A_\alpha(\theta) \to A_\alpha(\theta) - i\partial_\alpha\varphi(\theta). \tag{5.a}$$

Note that in an appropriately chosen gauge $A_\alpha(\theta)$ takes the form

$$A_\alpha(\theta) = \theta^\beta p_{\beta\alpha} + i\theta^\beta\theta_\beta c_\alpha,$$

with $p_{\alpha\beta} = p_{\beta\alpha} = i\gamma^m_{\alpha\beta}p_m$, where p_m is an even vector.

If one considers $A_m(x)$ and $A_\alpha(\theta)$ to be free fields then they satisfy

Free Maxwell equations.

$$\partial_m F^{mn}(x) = 0, \tag{6}$$

where $F^{mn}(x) = \partial_m A_n(x) - \partial_n A_m(x)$.

$$\partial_\alpha F^{\alpha\beta}(\theta) = 0, \tag{6a}$$

where $F^{\alpha\beta}(\theta) = \partial_\alpha A_\beta(\theta) + \partial_\beta A_\alpha(\theta) = p_{\alpha\beta} + p_{\beta\alpha} + 2i(\theta_\alpha c_\beta + \theta_\beta c_\alpha)$. So $A_\alpha(\theta)$ with $c_\alpha = 0$ is the solution of Eq. (6.a). Substituting this solution into Eq. (4.a) and choosing the gauge mentioned above we get the equation

$$L^\alpha \mathrm{D}_\alpha \Phi(\theta) = 0, \tag{4.b}$$

where $\mathrm{D}_\alpha = \frac{\partial}{\partial\theta^\alpha} + \theta^\beta p_{\beta\alpha}$ one may recognize the supercovariant derivative of the $N = 1$, $D = 2 + 1$ SUSY theory [9] in the momentum representation with the role of the momentum being plaid by the corresponding A_α component.

The integrability conditions for Eqs. (4) and (4.b) are

$$D_m D^m \psi(x) + \frac{i}{2} F_{mn} \gamma^m \gamma^n \psi(x) = 0. \tag{7}$$

The second term in (7) describes the electromagnetic interaction caused by the nonzero fermion magnetic moment.

$$\mathrm{D}_\alpha \mathrm{D}^\alpha \Phi(\theta) - 2ip_{\alpha\beta} L^\alpha L^\beta \Phi(\theta) = 0. \tag{7.a}$$

The analogous term is present in Eq. (7.a), but now, since we treat $p_{\alpha\beta}$ as the $D = 2+1$ semion momentum and in view of Eq. (2.a), $p_{\alpha\beta} L^\alpha L^\beta$ is interpreted as the Pauli–Lubanski Casimir operator of the $D = 2 + 1$ (super)Puancaré group. If $\Phi(\theta)$ is an eigenfunction of this operator, Eq. (7.a) splits into

$$p_{\alpha\beta} L^\alpha L^\beta \Phi(\theta) = m\Phi(\theta)$$

and

$$(\mathrm{D}_\alpha \mathrm{D}^\alpha - 2im)\Phi(\theta) = 0, \tag{7.b}$$

describing, in the momentum representation, the propagation of a free $N = 1, D = 2 + 1$ SUSY semion with mass m and the lowest superhelicity $1/4$ (at this $p_{\alpha\beta}$ satisfies the mass-shell condition $p_{\alpha\beta} p^{\alpha\beta} = 2m^2$) [1].

Note that if we took into account a "massive" term in Eq. (4.b), that is drew an analogy with the massive Dirac equation $(i\gamma^m \partial_m - m)\psi(x) = 0$, then we would get a semion superfield possessing an arbitrary superhelicity $\frac{1}{2}(\frac{1}{2}+n)$ $(n = 0, 1 \ldots \infty)$, with the "mass" parameter being proportional to the superhelicity value.

Without going into further details we just assert that to construct a semion action from which Eqs. (7.b) arise one should take into account gravity in the Grassmann space [10], and it is just here the analogy with the DME theory reaches its "top".

Dirac-Maxwell-Einstein action.

$$S = \int d^3x \sqrt{g} (\psi \gamma^{(m)} e^n_{(m)} \hat{D}_n \psi - (\frac{1}{4} F_{mn} F^{mn} + R(x) + \lambda)), \tag{8}$$

where $g_{mn}(x)$ is a Riemann metric, $\sqrt{g} \equiv \det(g_{mn}(x))$; $e^n_{(m)}$ is the veilbein ((m) corresponds to the local tangent space), $\hat{D}_n$ contains the spin connection; $R(x)$ is the Riemann scalar curvature and λ is a cosmological constant.

Semion action.

$$S = \int d^2\theta G(\theta)(i\Phi^\dagger L^{(\alpha)} e^{\beta}_{(\alpha)} \hat{D}_\beta \Phi - (\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + R(\theta) - 6m^2)), \qquad (8.a)$$

where $g_{\alpha\beta}(\theta) = \frac{1}{G(\theta)}\varepsilon_{\alpha\beta}$ is a Riemann metric of the Grassmann space, $e^{\beta}_{(\alpha)}$ is the corresponding veilbein ((α) corresponds to the local tangent space) $\hat{D}_\alpha$ contains the spin connection, $R(\theta) = 3i\partial_\alpha \partial^\alpha G(\theta)$ is the Riemann scalar curvature and $6m^2$ is a cosmological constant (determining the semion mass) whose sign is fixed by the requirement for $p_{\alpha\beta}$ to be time-like (the absence of tacheons). Note that the action is even since L^α changes the statistics of the function it is acting on (see above), i.e. effectively anti-commutes with D_α. At this $\Phi(\theta)$ is a representation of a supergroup realized on L^α rather than the representation of the Heisenberg–Weyl group.

We assume that Eq. (8.a) may be considered as the action of the effective $D = 2 + 1$, $N = 1$ SUSY semion model in the momentum representation and that the coordinate representation may be obtained by a functional integration of (8.a) over all independent configurations of the gauge field.

The semion model proposed is essentially based on the analogy with and uses the principles of the construction of the DME theory, which allows one to provide all fields of the model with clear gauge and geometrical meaning. This seems to be useful for the development of the field-theoretical approach to anyons [11,4,1], which, in particular, encounters problems in constructing equations of motion and actions due to the absence of a reliable geometrical and symmetry basis.

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