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HIGH FREQUENCY STOCHASTIC RESONANCE IN PERIODICALLY DRIVEN SYSTEMS

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High frequency stochastic resonance (SR) phenomena, associated with fluctuational transitions between coexisting periodic attractors, have been investigated experimentally in an electronic model of a single-well Duffing oscillator bistable in a nearly resonant field of frequency ω_F. It is shown that, with increasing noise intensity, the signal/noise ratio (SNR) for a signal due to a weak trial force of frequency $\Omega \sim \omega_F$ at first decreases, then *increases*, and finally decreases again at higher noise intensities: behaviour similar to that observed previously for conventional (low frequency) SR in systems with static bistable potentials. The stochastic enhancement of the SNR of an additional signal at the mirror-reflected frequency $|\Omega - 2\omega_F|$ is also observed, in accordance with theoretical predictions. Relationships with phenomena in nonlinear optics are discussed.

The first decade of research on *stochastic resonance* [1] (SR), in which the signal due to a weak (trial) periodic force in a nonlinear system can be optimally amplified by the introduction of external noise of appropriate intensity, has concentrated almost exclusively on systems with coexisting attractors corresponding to the minima of a symmetrical, static, bistable potential. In such cases, the noise-induced amplification arises through the occurence of fluctuational transitions between the attractors. For suitably chosen noise intensity, these become nearly periodic at the frequency of the trial force, with an average amplitude that can approach the half-separation of the attractors. This is the mechanism responsible for SR phenomena studied in connection with ice-ages, ring-lasers, electronic circuits, passive optical systems, electron spin resonance, sensory neurons, a magnetoelastic ribbon, and a laser with saturable absorber; we shall refer to it as conventional SR.

More recently, efforts have been initiated to seek SR in other classes of systems having quite different kinds of attractors: it has now been identified in underdamped nonlinear oscillators that have single static attractors [2], in a bistable, chaotic, electrical circuit with two coexisting attractors of which one is a limit cycle and the other is chaotic [3], and in a system with two coexisting stable limit cycles that have the *same period* and correspond to periodically forced vibrations of a damped nonlinear oscillator [4]; the onset of a resonant absorption

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that increased extremely sharply with noise intensity - a phenomenon that has much in common with SR - had earlier been demonstrated theoretically [5] for the latter system. In this Letter we shall show that SR of this latter kind, while similar to conventional SR in some respects, also has a number of interesting features that distinguish it from earlier manifestations of the phenomenon and which have implications for four-wave mixing in nonlinear optics. We expect the ideas to be applicable to a large class of passive optically bistable systems and, in particular, to optically bistable microcavities [6].

The system that we consider is the nearly-resonantly-driven, underdamped, single-well Duffing oscillator with additive noise,

$$\ddot{q} + 2\Gamma\dot{q} + \omega_0^2 q + \gamma q^3 = F\cos(\omega_F t) + f(t), \tag{1}$$

$$\Gamma, |\delta\omega| \ll \omega_F, \qquad \gamma\delta\omega > 0, \qquad \delta\omega = \omega_F - \omega_0,$$

$$\langle f(t)\rangle = 0, \qquad \langle f(t)f(t')\rangle = 4\Gamma T\delta(t-t').$$

This system is of topical interest on account of its importance in nonlinear optics [7] and its relevance to experiments on a confined relativistic electron excited by cyclotron resonant radiation [8]. Provided that $F^2 \ll \omega_0^4(\delta\omega^2 + \Gamma^2)/|\gamma|$, and that the noise is weak, the resultant comparatively small amplitude $[\ll (\omega_0^2/|\gamma|)^{\frac{1}{2}}]$ oscillations of $q(t)$ can conveniently be discussed in terms of the dimensionless parameters [5]:

$$\eta = \Gamma/|\delta\omega|, \qquad \beta = \frac{3|\gamma|F^2}{32\omega_F^3(|\delta\omega|)^3}, \quad \alpha = 3|\gamma|T/8\omega_F^3\Gamma, \tag{2}$$

which characterise, respectively, the frequency detuning, the strength of the main periodic field, and the noise intensity. The bistability [9] in which we are interested arises for a restricted range of η and β: within the triangular region bounded by the full lines of Fig.1. Thus, as the amplitude of the periodic force is gradually increased from a small value at fixed frequency (see vertical line a - a′), the system moves from monostability (one small limit cycle), to bistability (two possible limit cycles of different radii), and then back again to monostability (one large limit cycle). The effect [5] of weak noise $f(t)$ is to cause small vibrations about the attractors, and to induce occasional transitions (cf [10]) between them when the system is within the bistable regime. We shall see that SR phenomena occur in the close vicinity of the kinetic phase transition (KPT) line [11], shown dashed in Fig 1, where the populations of the two attractors are equal.

Our principal aim is to consider the response of the system (1) to an additional weak trial force $A\cos(\Omega t + \phi)$. The combined effects of dissipation and noise result in a steady statistical distribution, and the response of the system can therefore be described, in terms of linear response theory, by a susceptibility. The trial force beats with the main periodic force and thus gives rise to vibrations of the system, not only at Ω, but also at the combination frequencies $|\Omega \pm 2n\omega_F|$ (and also at $|\Omega \pm (2n+1)\omega_F|$ in the case of nonlinearity of a general type). We are interested in the case where the strong and trial forces are both nearly resonant: that is, ω_F and Ω both lie close to ω_0. This is the case for which the response to the trial force is strongest. It is at its most pronounced at frequency Ω and at the nearest

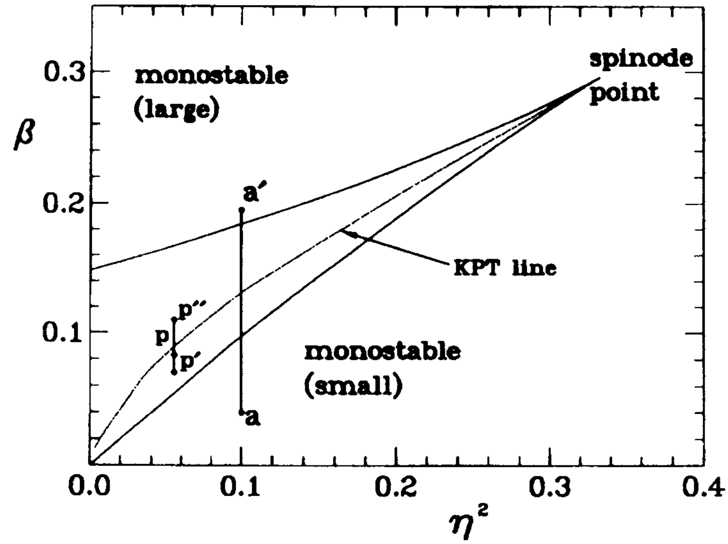

Fig.1. Phase diagram for (1) in terms of reduced parameters (2): the cuts a-a', p'-p-p'' are discussed in the text

resonant combination, which for (1) is $|\Omega - 2\omega_F|$. The amplitudes of vibrations at these frequencies can be described respectively by susceptibilities $\chi(\Omega)$, $X(\Omega)$, so that the trial-force-induced modification of the coordinate q, averaged over noise, can be sought in the form

$$\delta\langle q(t)\rangle = A\mathrm{Re}\{\chi(\Omega)\exp[-i\Omega t - i\phi] + X(\Omega)\exp[i(2\omega_F - \Omega)t] - i\phi]\}. \tag{3}$$

Within the KPT range, for Ω close to ω_F, $|\mathrm{Im}\chi(\Omega)|$ becomes large and strongly noise-dependent [5]. The rapid rise in susceptibility with noise intensity corresponds precisely to SR since, according to (3), the squared amplitudes of the signals at frequencies Ω and $|\Omega - 2\omega_F|$ (and the integrated powers of the corresponding peaks in the power spectrum) are

$$S(\Omega) = \frac{1}{4}A^2|\chi(\Omega)|^2, \quad S(|\Omega - 2\omega_F|) = \frac{1}{4}A^2|X(\Omega)|^2. \tag{4}$$

An intuitive understanding of the mechanism of stochastic amplification can be gained by noting that the trial force modulates the driving force (and the coordinate $q(t)$) at frequency $|\Omega - \omega_F|$ and that, when $|\Omega - \omega_F|$ is small, the system responds almost adiabatically. In terms of the phase diagram Fig.1, the beat envelope then results in a slow vertical oscillation of the operating point. If the operating point is taken to be p, and its range of modulation $\mathrm{p}' - \mathrm{p}''$ is set to straddle the KPT line as shown, and the noise intensity is optimally chosen, then the system will have a tendency to make inter-attractor transitions *coherently*, once per half-cycle of the beat frequency. The net effect of the noise is therefore to increase the modulation depth of the beat envelope of the response, thereby increasing the components of the signal at frequencies Ω and $|\Omega - 2\omega_F|$.

We are investigating the response of the system (1), and the variation of the signal/noise ratio with α, through analogue experiments on an electronic model of

(1), of which the relevant technical details were given in a recent conference report [4]. In terms of scaled units the circuit parameters were set, typically, to: 2Γ = 0.0397; ω_0 = 1.00; γ = 0.1; ω_F = 1.07200; Ω = 1.07097; and, to seek SR near the KPT, F = 0.068 and the amplitude of the trial force A = 0.006. A spectral density of fluctuations of the coordinate $q(t)$ (about $\langle q(t)\rangle$ for A=0) recorded with the above parameter values for α = 0.061 and 16384 samples in each realization, is shown in Fig 2. The smooth background is the supernarrow peak of [11], here broadened by noise (although its width remains very much smaller than 2Γ); delta function spikes, indicated by raised points [12] of the discrete spectrum, are clearly visible, not only at the trial force frequency (Ω), but also as predicted at the mirror-reflected frequency ($2\omega_F - \Omega$).

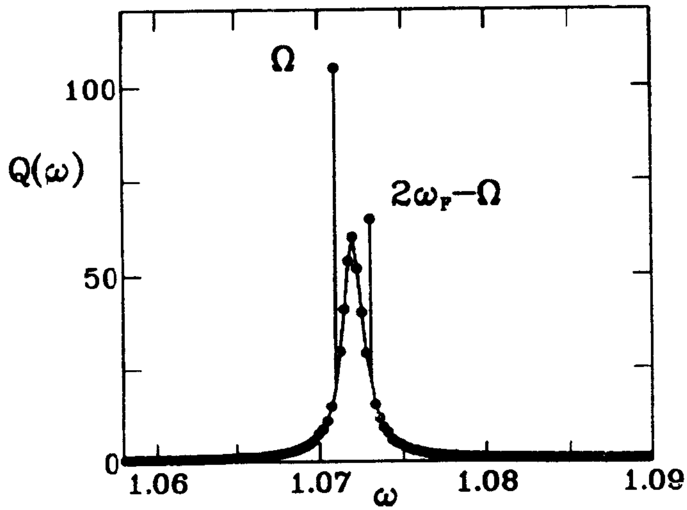

Fig.2. Power spectral spectral density $Q(\omega)$ measured for the electronic model, with the contents of each FFT memory address shown as a separate data point on a highly expanded abscissa; a smooth curve, peaking at ω_F, is drawn through the background spectrum; vertical lines indicate the delta spikes resulting from the trial force

The signal/noise ratios R, determined in the usual way [1] from measurements of the delta spikes and the smooth background, are plotted (data points) as functions of noise intensity α in Fig.3 for β = 0.814, η = 0.236. It is immediately evident that there is a range of α within which R *increases* with α. It is also apparent that, for both the main and the mirror-reflected signals, the form of $R(\alpha)$ in Fig 3 is remarkably similar to that observed earlier [1] in the case of conventional SR. That is, R initially decreases with α, on account of the increase in its denominator. With further increase of α, the inter-attractor transitions come into play and become phase-coherent with the trial force to a high probability, giving rise to an increase in R through the stochastic amplification mechanism discussed above. Finally, for still larger α, R decreases again partly owing to the continuing rise in its denominator and partly because transitions are then occurring very frequently, within individual periods of the trial force, with a corresponding reduction in the proportion of the phase-coherent jumps that are responsible for the amplification.

A quantitative theory of the phenomenon is readily constructed through an extension [13] of [5]. It leads to contributions to the susceptibilities from inter-attractor transitions of the form

$$\chi_{tr}(\Omega) = \frac{w_1 w_2}{2\omega_F(\omega_F - \omega_0)}(u_1^* - u_2^*)\frac{\mu_1 - \mu_2}{\alpha}\left[1 - \frac{i(\Omega - \omega_F)}{W_{12} + W_{21}}\right]^{-1}, \tag{5}$$

$$X_{tr}(\Omega) = \frac{u_1 - u_2}{u_1^* - u_2^*}\chi_{tr}(\Omega), \quad \mu_j = \sqrt{\beta}(\frac{\partial R_j}{\partial \beta}),$$

where w_1, w_2 are the populations of the attractors 1, 2 and W_{12}, W_{21} are the probabilities of transitions between them, which are of the activation type $W_{ij} \propto \exp(-R_i/\alpha)$. (The u_i, u_i^*, which define the positions of the attractors in the rotating coordinate frame, can be regarded as constants for given η, β). It is evident that the contributions (5) come into play if, and only if, the system is within the KPT range where the populations of the unperturbed attractors are comparable: otherwise, the factor $w_1 w_2 \propto \exp(-|R_1 - R_2|/\alpha)$ will be exponentially small. Within the KPT range, however, the susceptibilities will be large because they are proportional to the large factor $|\mu_1 - \mu_2|/\alpha$; the rapid increase of W_{ij} with noise intensity then ensures that there will be a range of α in which both susceptibilities increase very rapidly with α, consistent with the experiments. The full theory [13], including the effect of intra-attractor vibrations, leads to the curves of Fig.3. Given the large systematic errors inherent in the measurements - arising e.g. from $\delta\omega$ (1), a small difference between large quantities which, in β (2), is then raised to its third power - the agreement between theory and experiment can be considered excellent.

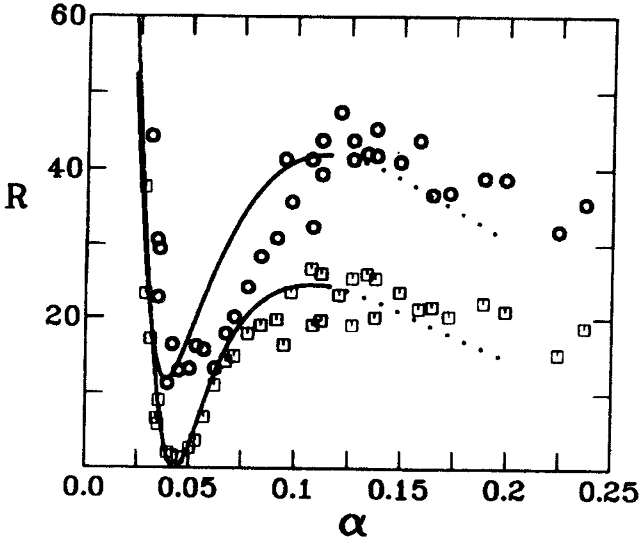

Fig.3. The signal/noise ratio R of the response of the system (1) to a weak trial force at frequency Ω, as a function of noise intensity α, in experiment and theory: at the trial frequency Ω (circle data and associated theoretical curve); and at the "mirror-reflected" frequency $(2\omega_F - \Omega)$ (squares). For noise intensities near those of the maxima in $R(\alpha)$, the asymptotic theory is only qualitative and so the curves are shown dotted

In conclusion, we would emphasize, first, that, in contrast to earlier forms of bistable SR [1, 3], stochastic amplification occurs here *not* at the relatively low frequency of the quasi-periodic inter-attractor hopping but, rather, at Ω close to the much higher (tunable) frequency ω_F of the main periodic driving force. To emphasize the distinction, it seems appropriate to refer to the new form of SR as high frequency stochastic resonance (HFSR).

Secondly, we draw attention to the relationship of HFSR to four-wave mixing in nonlinear optics [14]. In effect, our results correspond to noise-enhanced amplification of the signal wave, and noise-enhanced generation of the idler wave. The mechanisms are *resonant* and, although they have the appearance of four-wave mixing, they actually correspond to multiple-wave processes: in terms of quantum optics, the oscillator absorbs and re-emits many quanta of the strong field. The very high amplification/generation coefficients arise partly from their resonant character and partly from the fact that the signal sizes correspond, not to the amplitudes of vibrations about the attractors but, as usual in bistable SR, to the "distance" between the attractors (to their coordinate separation for conventional SR, and approximately to their difference in amplitude in the present case).

Finally, our prediction and demonstration of HFSR for periodic attractors, and its similarity (Fig 3) to conventional SR, leads to a broader and more general perception of the physical nature of bistable SR. Like the onset of supernarrow peaks in the power spectra, conventional SR [1] and HFSR are both critical phenomena that arise in the range of the KPT. While HFSR is to be anticipated for coexisting stable limit cycles with equal periods, low-frequency SR is a more robust effect. It arises, not only for systems fluctuating in simple double-well potentials but also for systems where one (or both) of the attractors is chaotic [3]; we may infer from [5] that, in general, low frequency SR is also to be anticipated for periodic attractors [although not for (1), where the centres of the forced vibrations are independent of amplitude]. Since noise gives rise to fluctuational hopping between any type(s) of attractors, it seems reasonable to conclude that SR is actually a quite general phenomenon characteristic of *all* systems with coexisting attractors, regardless of the nature of those attractors, provided only that the system lies within its KPT range.

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1. For a recent review, see J. Stat. Phys. **70**, N° 1/2 (1993), special issue on stochastic resonance.
2. N.G.Stocks, N.D.Stein, S.M.Soskin, and P.V.E.McClintock, J Phys A **25**, L1119 (1992); N.G.Stocks, N.D.Stein, and P.V.E.McClintock, J Phys A **26**, L385 (1993).
3. V.S.Anishchenko, M.A.Safonova, and L.O.Chua, Int. J. of Bifurcation and Chaos **2**, 397 (1992).
4. M.I.Dykman, D.G.Luchinsky, R.Mannella, et al., J. Stat. Phys. **70**, 479 (1993).
5. M.I.Dykman and M.A.Krivoglaz, Sov. Phys. J.E.T.P. **50**, 30 (1979).
6. H.J.Carmichael, in Optical Instabilities, eds. R.W.Boyd, M.G.Raymer, and L.M.Narducci, Cambridge University Press, 1986, p.111.
7. H.M.Gibbs, Optical Bistability: Controlling Light with Light, (Academic Press, New-York, 1985); P.D.Drummond and D.F.Walls, J. Phys. A **13**, 725 (1980); Chr. Flytzanis and C.L.Tang, Phys. Rev. Lett. **45**, 441 (1980); J.A.Goldstone and E.Garmire, Phys. Rev. Lett. **53**, 910 (1984).
8. G.Gabrielse, H.Dehmelt, and W.Kells, Phys. Rev. Lett. **54**, 537 (1985).
9. L.D.Landau and E.M.Lifshitz, Mechanics (Pergamon, London, 1976).
10. M.R.Beasley, D.D'Humieres, and B.A.Huberman, Phys. Rev. Lett. **50**, 1328 (1983).
11. M.I.Dykman, R.Mannella, P.V.E.McClintock, and N.G.Stocks, Phys. Rev. Lett. **65**, 48 (1990).

12. The value of Ω, the number of points in each $q(t)$ realization, and the sample interval, were chosen to be such that the signals at Ω and $(2\omega_F - \Omega)$ each fell within individual bins of the discrete SDF, and such that there were several bins in between them.
13. M.I.Dykman, D.G.Luchinsky, R.Mannella, et al., in preparation.
14. Y.R.Shen, The Principles of Nonlinear Optics (Wiley, New-York, 1984).