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LOCAL AND GLOBAL QUANTUM EFFECTS ON COSMIC STRING SPACE-TIME

D.V. Fursaev [1)]

Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Head Post Office
P.O.Box 79, Moscow, Russia

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The effective action and vacuum energy of a quantum scalar field around a cosmic string are considered. Analogy is pointed out with quantum theory with boundaries. The surface infinities in the effective action are shown to appear and are removed by renormalization of the string tension. Besides, the total renormalized energy of the string and field turns out to be finite due to cancelation of the known non-integrable divergence in the energy density of the field with a counterterm in the bare string tension.

1. Quantum field theory on cosmic string space-time has attracted much attention in the last years [1–6], because of its possible relevance in cosmology [7]. The space near an idealized cosmic string of zero thickness possesses interesting properties. It looks as a conical space where the line element can be written in a form like on the plane in polar coordinates, $ds^2 = dr^2 + r^2 d\varphi^2$, but with a polar angle φ ranging from 0 to a positive parameter $\alpha = 2\pi(1 - 4\mu G)$, where μ is the string tension [7–8]. Besides, this space can be considered as a space whose curvature is concentrated at the zero radius $r = 0$ and looks there like a delta function [8].

In connection with conical singularities the quantum theory on orbifold factors of the Riemannian manifolds [9–11] is worth to be mentioned as well. Singularities of that sort appear at fixed points of the corresponding isometry groups.

In the present paper we investigate the local and global effects of vacuum polarization around a cosmic string and demonstrate a close analogy with quantum theories on spaces with boundaries [12, 13]. The global effects are displayed in integral quantities like the ground energy or the effective action of the field.

[1)] e-mail: fursaev@theor.jinrc.dubna.su

They can be derived by using the trace of the heat kernel on a cone that was shown to look essentially different at asymptotically small values of the proper time as compared to the plane heat kernel [14]. For this reason, the effective action obtained on its base includes a surface divergent functional given on the string world sheet. It is interesting that these surface infinities can be removed by renormalization of the string tension rendering finite the total renormalized energy. The similar situation takes place in quantum theories with boundaries where divergent terms arise on boundary surfaces giving rise to renormalization of bare surface gravitational actions [12]. The analogy can be continued further to demonstrate that finiteness of the total renormalized energy results from cancellation of the non-integrable divergence in the energy density with a surface counterterm appearing from the bare string tension.

2. For simplicity we confine the following analysis to the case of a free scalar field of the mass m around an infinitely thin straight string that is at rest along the z axis. The metric around it can be written in the form $ds^2 = dt^2 - dz^2 - dr^2 - r^2 d\varphi^2,\ \ 0 \leq \varphi \leq \alpha$.

The vacuum energy $E_0(\alpha)$ of the field can be calculated in two ways. The first one is to obtain $E_0(\alpha)$ as the integral of the renormalized energy density $< \hat{T}_{00}(x) >^{\alpha}_{sub}$. However this method cannot be applied immediately so far as the renormalized energy momentum tensor has a non-integrable infinity at the string axis [3–5] and an additional regularization will be shown below to be needed. We consider at first another way based on the thermodynamical relation between the internal energy $E_{\beta^{-1}}$ of the system at a temperature β^{-1} and the partition function Z_β

$$E_{\beta^{-1}} = < \hat{H} >_\beta = -\frac{\partial}{\partial\beta} \log Z_\beta, \quad Z_\beta = Tr(e^{-\beta\hat{H}}) \tag{1}$$

where $\hat{H}$ is the Hamiltonian. In such approach the ground energy E_0 is the energy at zero temperature $E_0 = \lim_{\beta\to\infty} E_{\beta^{-1}}$ and it can be derived without using the renormalized energy density.

Another quantity we are interested in is the effective action W that can be also defined for a finite temperature with the help of the partition function

$$W = -\log Z_\beta, \tag{2}$$

by taking next the limit $\beta \to \infty$. Its variations coincide with the thermal average of the variations of the functional

$$S_e[\phi] = \int \sqrt{g} d^4x 1/2(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2)$$

that are interpreted as quantum operators

$$\delta W = Z_\beta^{-1} \int D\phi \ \ \delta S_e[\phi] e^{-S_e[\phi]} = < \delta\hat{S}_e >_\beta, \quad Z_\beta = \int \ D\phi e^{-S_e[\phi]}. \tag{3}$$

We use in (3) the fact that the partition function can be represented in the form of a functional integral by passing to a periodic imaginary time and consequently the action $S_e[\phi]$ is taken here on the Euclidean section of the string space-time $ds^2 = d\tau^2 + dz^2 + dr^2 + r^2 d\varphi^2,\ \ 0 \leq \tau \leq \beta$. Thus W is an Euclidean form of the effective action but transition to the convenient definition [13, 15] doesn't make any difficulty for the static space we consider.

If we represent the action W in the form

$$W = \frac{1}{2}\log\det(\Delta + m^2) = \frac{1}{2}Tr\log(\Delta + m^2) = -\frac{1}{2}\int_0^\infty \frac{ds}{s} Tr\left(e^{-s\Delta}\right)e^{-m^2 s}, \quad (4)$$

where Δ is the corresponding Laplace operator, then the structure of its ultraviolet divergences follows immediately from the asymptotic expansion as $s \to 0$ of the trace $Tr\left(e^{-s\Delta}\right)$ of the heat kernel on the Euclidean string space-time [14]

$$Tr\left(e^{-s\Delta}\right) = \frac{1}{(4\pi s)^2}\left(\Omega + \Sigma\alpha C_1(\alpha)\ s\right) + ES, \qquad (\beta \to \infty) \quad , \quad (5)$$

where $C_1(\alpha) = \frac{1}{6}\left((2\pi\alpha^{-1})^2 - 1\right)$, ES means the terms that vanish exponentially as $s \to 0$ and Ω is the volume of the space. The second term in (5) is provided only by the conical singularities on the string and it is proportional so to the area Σ of the surface $r = 0$, the string world sheet. It is important that this term would not appear, if the integration over the space-time in the effective action were stopped short before the point $r = 0$, by no matter how close.

So far as the space is non-compact, Ω and Σ are to be infinite and thus (5) has to be treated in a regularized sense. In this case the ES terms are significant. However, if L is a typical size of the space (the length at which the integrals are cut off), then ES terms in (5) can be shown to be of the order $s^{-1}\exp(-L^2/s)$. Therefore from now on we can drop ES as negligible in the limit $L \to \infty$ we are interested in.

After substitution (5) into (4) one can represent the effective action as a sum of the volume and surface parts proprtional to Ω and Σ respectively: $W = W_{vol} + W_{surf}$. The volume term W_{vol} in the action turns out to be exactly the same as in Minkowsky space and it develops standard divergences appearing at the lower integration limit in (4) as $s \to 0$. Besides, the conical singularities at the string axis result to an additional divergent surface term W_{surf}, the exact form of which is not relevant for this analysis. Since the latter is proportional to the area Σ of the string world sheet, it is worth to unify this term with the string action that can be added to W and remove the divergence by a renormalization of the bare string tension μ_B. With the help of this recipe one can get the total renormalized effective action

$$W_{tot} = W + \mu_B\Sigma = W_{vol} + \mu\Sigma \quad (6)$$

where the infinities in W_{vol} are to be removed by a standard procedure and the string action is expressed through the renormalized tension μ. (In (6) we used the fact that W and W_{tot} are the actions on the space-time with an Euclidean signature resulting to the string action in the form $\mu\Sigma$ where $\Sigma = \beta\int dz$ and $\beta \to \infty$.)

By taking into account (1), (2), on can write the vacuum energy as $E_0(\alpha) = \partial/\partial\beta W$ (as $\beta \to \infty$). However W is a divergent functional. Consequently, a finite result can be obtained for the total energy $E_{tot} = \partial/\partial\beta W_{tot}$ that is defined by the renormalized total action and includes the energy of the string. In this way, after subtracting from E_{tot} the vacuum energy in the Minkowsky space, that is equivalent to omitting the volume part W_{vol} in W_{tot}, we have

$$E_{tot} = \mu\int dz \quad (7)$$

It shows that the total renormalized energy per unite length turns out to be finite and determined only by the renormalized string tension μ.

3. Until now we dealt with the integral quantities like the effective action and total energy using for their calculation the trace of the heat kernel. The surface terms appearing in the functional W have a global origin: they would have not arisen, if we had excluded, from the integrals over the space-time, the region around the string world sheet. The local renormalized energy momentum tensor near the cosmic string was calculated by a number of authors [3–5]. Let us find out a connection between their and our results and demonstrate that the local non-integrable divergence in the average energy density arising as the string is approached can be removed by a suitable renormalization of the bare string tension so that the total energy turns out to be finite. What we are going to do is to explore the same approach as used in [12] in quantum theory with boundaries.

We consider a real massless scalar field for which the energy density can be obtained in the closed form [3]

$$< \hat{T}_{00}(x) >^{\alpha}_{sub} = \frac{1}{16\pi^2 r^4} (2(1-4\xi)C_1(\alpha) - C_2(\alpha)) \tag{8}$$

where $C_2(\alpha) = 1/90\left((2\pi\alpha^{-1})^2 - 1\right)\left((2\pi\alpha^{-1})^2 + 11\right)$ The value $\xi = 1/6$ corresponds to a conformally invariant field. The local energy is evaluated in a standard way from the Green function $G^{\alpha}(x,x') = i^{-1} < T(\hat{\phi}(x), \hat{\phi}(x')) >$ as a coincidence limit $< \hat{T}_{00}(x) >^{\alpha}_{sub} = \lim_{x'\to x} \mathcal{T}_{00} G^{\alpha}_{sub}(x,x')$ where $\mathcal{T}_{00}$ denotes a second order differential operator [16] depending on the type of field and the divergences are removed as ordinary by subtracting the Minkowsky Green function $G^{\alpha=2\pi}$ in G^{α}_{sub}.

It is obvious that the local divergence of the energy density (8) at $r=0$ can be regularized if we restrict the domain of integration in the total energy by the values of coordinates $r \geq r_0$ where r_0 is a positive small parameter that can be treated as the string radius. Moreover the regularization suggested also makes finite the surface term in the effective action. This can be demonstrated for the particular values of the parameter $\alpha = 2\pi n^{-1}$, $n = 2, 3, ..$, when the heat kernel $K_\alpha(r, r', \varphi - \varphi')$ on the conical space can be represented in a closed form with the help of the plane heat kernel $K(r,r',\varphi-\varphi') = (4\pi s)^{-1}\exp\left(-(r^2+r'^2-2rr'\cos(\varphi-\varphi'))/4s\right)$. For instance, for $\alpha=\pi$ one can write $K_\alpha(r,r',\Delta\varphi) = K(r,r',\Delta\varphi) + K(r,r',\Delta\varphi+\pi)$. In this case the "regularized" trace is easily calculated and we get instead of (5)

$$Tr(e^{-s\Delta})_{r_0} \equiv \int_{r_0}^{\infty} r dr \int_0^{\alpha} d\varphi \int d\tau dz \frac{1}{4\pi s} K_\alpha(r,r,0) = \frac{1}{4\pi s}\left(\Omega + \frac{\pi s}{2} e^{-r_0^2/s}\Sigma\right) \tag{9}$$

Now the surface term in the trace doesn't produce a divergence in the effective action (4) after integration over s. The total effective action can be defined as before and takes the form

$$W_{tot} = \frac{1}{2}\log\det(\triangle + m^2)_{r_0} + \mu_B\Sigma = -\frac{1}{2}\int_0^{\infty}\frac{ds}{s} Tr\left(e^{-s\Delta}\right)_{r_0} e^{-m^2 s} + \mu_B\Sigma \equiv$$

$$\equiv W_{vol} + W_{r_0,surf}. \tag{10}$$

It is separated into the volume part W_{vol} proportional to Ω and the surface part $W_{r_0,surf}$ given on the world sheet Σ. As distinct from W_{vol} developing the standard divergences, the surface action $W_{r_0,surf}$ turns out to be finite while $r_0 \neq 0$. In the case of arbitrary values of α and for zero mass its expression can be exactly obtained with the help of an integral representation [1, 2] of the kernel K_α

$$W_{r_0,surf} = \left(\mu_B - \frac{\alpha C_2(\alpha)}{32\pi^2 r_0^2}\right)\Sigma \quad . \tag{11}$$

It follows from (11) that the divergence in $W_{r_0,surf}$ at $r_0 \to 0$ can be removed by replacing as before the bare string tension μ_B by the renormalized μ

$$\mu_B = \mu + \frac{1}{32\pi^2 r_0^2}\alpha C_2(\alpha). \tag{12}$$

Taking this into account one can write the local renormalized energy as the sum

$$< \hat{T}_{00} >^\alpha_{r_0,ren} = T_{00,B} + < \hat{T}_{00} >^\alpha_{r_0,sub} \tag{13}$$

of the string energy $T_{00,B} = \mu_B \delta_2(r)$ concentrated at the string axis ($\delta_2(r)$ is the delta function on a cone

$$\int_0^\infty r dr \int_0^\alpha d\varphi \delta_2(r) = 1)$$

and the renormalized energy density $< \hat{T}_{00} >^\alpha_{r_0,sub}$ of the quantum field in the domain $r \geq r_0$. Two densities, $< \hat{T}_{00} >^\alpha_{sub}$ given by (8) and $< \hat{T}_{00} >^\alpha_{r_0,sub}$ coincide everywhere except the region near the string. To demonstrate this let us calculate the classical energy-momentum tensor of the field in this domain defined by the functional differentiation of the action that we take in the same form as in [12]

$$S = -\frac{1}{2}\int_{r \geq r_0} d^4x\sqrt{-g} \ \ \phi(x)[\Box + \xi R]\phi(x) \tag{14}$$

where $\Box = \sqrt{-g}^{\ -1}\partial_\mu \sqrt{-g} g^{\mu\nu}\partial_\nu$ is the D'Alambertian and R is the scalar curvature. The variation of this functional δS under changing the metric $\delta g^{\mu\nu}$ consists of two parts

$$\delta S = \frac{1}{2}\int_{r \geq r_0} d^4x\sqrt{-g} \ \ T_{\mu\nu}(x)\delta g^{\mu\nu}(x) + \delta_{surf} S \tag{15}$$

where $T_{\mu\nu}$ stands for the normal expression of the stress tensor of a scalar field [17] and an additional surface term arises due to the restriction of the domain of integration

$$\delta_{surf} S = -\frac{1}{2}\int_{r=r_0} d\sigma^r \left[\phi^2(g_{\mu\nu}\delta g^{\mu\nu}{}_{;r} + g_{r\mu}\delta g^{\mu\sigma}{}_{;\sigma}) + \right.$$
$$\left. + \left((1/4 - \xi)(\phi^2)_{,r}\, g_{\mu\nu} + (\xi - 1/2)(\phi^2)_{,\nu}\, g_{\mu r}\right)\delta g^{\mu\nu}\right] \tag{16}$$

($d\sigma^r$ is the area element). So far as there is no real boundary of the space on the surface $r = r_0$, the variations of the metric $\delta g^{\mu\nu}|_{r=r_0}$ don't vanish on it. They are independent of its normal derivatives on the surface and thus the last ones can be ignored. As a result, $\delta_{surf} S$ produces the additional term in the energy density

$$T_{00,surf} = \frac{2}{\sqrt{-g}}\frac{\delta_{surf} S}{\delta g^{00}} = (1/4 - \xi)\delta(r - r_0)\frac{d}{dr}(\phi)^2 \tag{17}$$

giving rise to the distinction between the average density in the domain, $<\hat{T}_{00}>^{\alpha}_{r_0,sub}$, and the local energy (8)

$$<\hat{T}_{00}>^{\alpha}_{r_0,sub}=<\hat{T}_{00}>^{\alpha}_{sub}+i(1/4-\xi)\delta(r-r_0)\lim_{x\to x'}\left(\frac{d}{dr}+\frac{d}{dr'}\right)G^{\alpha}_{sub}(x,x') \qquad (18)$$

($\delta(r-r_0)$ is the one-sided delta-function). For its calculation the proper-time representation for the Green function can be used and the final result reads [14]

$$<\hat{T}_{00}>^{\alpha}_{r_0,sub}=<\hat{T}_{00}>^{\alpha}_{sub}-(1/4-\xi)\frac{C_1(\alpha)}{4\pi r_0^3}\delta(r-r_0). \qquad (19)$$

Integrating now the renormalized quantity (13) over the space

$$E_{tot}=\int<\hat{T}_{00}>^{\alpha}_{r_0,ren}\ dv=\left[\mu_B+\int_{r_0}^{\infty}rdr\int_0^{\alpha}d\varphi<\hat{T}_{00}>^{\alpha}_{r_0,sub}\right]\int dz \qquad (20)$$

and using (12), (20) we find that the counterterm in the bare tension μ_B cancels exactly the term proportional to r_0^{-2} in the integrated energy of the field rendering finite the renormalized total energy as $r_0\to 0$

$$E_{tot}=\mu\int dz. \qquad (21)$$

This expression coincides with the total energy (7) derived before in another way from W_{tot}.

4. In this work a close analogy between quantum theory on the space with conical singularities and quantum theory with boundaries was outlined. In both cases the one loop quantum corrections result to divergent surface functionals in the effective actions. The renormalization of these functionals can be used to remove non-integrable divergence in the energy density and to obtain the finite total energy of the system. However, this analysis concerns the idealized objects, strings and boundaries of zero thickness. In effect one might expect that for the real string of a finite size the divergent terms on its world sheet give large but still finite contributions to the renormalized energy.

In the theory with boundaries the surface actions are known to essentially depend on which of the boundary conditions, Dirichlet or Neumann, are imposed. As for the string case, equation (5) implies a finite boundary condition on the string axis [14] and other possibilities are worth to be investigated as well. For example, the possibility of logarithmically divergent conditions has been pointed out in [6] in connection with the self-adjoint extensions of the Laplace operator on a cone.

It is to be also mentioned that our consideration was virtually confined to conical singularities in the flat space and incorporation of the curvature effects represents an interesting problem.

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