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NONLINEAR KINK OSCILLATIONS OF A MAGNETIC FLUX TUBE

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The nonlinear equations which describe the long wavelength, weakly dispersive kink oscillations propagating along a magnetic flux tube are derived. The character of nonlinearity appeared to be a cubic one with the coefficients which reflect the influence of a magnetic free environment on a transverse oscillations of flux tube.

In recent years the features of magnetic flux tubes are extensively studied because of their dominant role in dynamics of solar atmosphere: according the observational data magnetic field at solar surface occurs not in a diffuse form but it is concentrated in thin intense magnetic flux bundles embedded in almost nonmagnetized plasma. Usually magnetic flux tubes are isolated and far removed from each other, covering 90% of solar atmosphere outside sunspots [1]. In sunspot regions magnetic flux tubes form a dense conglomerate [2]. The structured magnetic field is often met also in laboratory plasma as well as in other astrophysical objects.

The interaction of magnetic flux tubes with the surrounding plasma results in the excitation of the different kind of oscillations propagating along flux tubes [3]. Among them the most important one is a kink oscillation corresponding to the dipole mode with the azimuthal wave number $m = \pm 1$ and a phase velocity

$$c = \frac{B}{\sqrt{4\pi(\rho_i + \rho_e)}}. \tag{1}$$

Here ρ_i and ρ_e are plasma densities inside and outside flux tube, and B is a magnetic field strength. For the time being the linear oscillations of flux tube are well understood, and are available elsewhere ([3], see also [4] and References therein).

In the present paper we derive the equations which govern the propagation of weakly nonlinear long wavelength kink mode propagating along magnetic flux tube surrounded by nonmagnetized plasma. We adopt the model of a cylindrical flux tube of radius R, which assumed to be much less than the wavelength $\lambda = k^{-1}$:

$$kR \ll 1. \tag{2}$$

The discussion is based on the ideal MHD equations which are written here for convenience:

$$\rho\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\vec{v} = -\nabla p + \frac{1}{4\pi}[\mathrm{rot}\mathbf{B}, \mathbf{B}], \tag{3}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{rot}[\mathbf{v}\mathbf{B}], \quad \frac{\partial p}{\partial t} + \mathrm{div}\rho\vec{v} = 0, \quad p = p_0\left(\frac{\rho}{\rho_0}\right)^\gamma. \tag{4}$$

The above set should be completed by the equation of a pressure balance in an equilibrium state of flux tube:

$$p_i(r) + \frac{B^2(r)}{8\pi} = p_e. \tag{5}$$

Here p_i and p_e are gas kinetic pressures inside and outside magnetic flux tube. For simplicity we assume that plasma inside flux tube is cold, $p_i \ll p_e$, and, respectively, neglect the gas-kinetic pressure inside flux tube. This assumption is not principal but allows us to make algebra not so long. The set (3), (4) describes the motions both inside flux tube and outside it (where this set becomes pure hydrodynamic one). At the surface of flux tube the boundary conditions of continuity of the normal component of velocity,

$$v_{ri} \mid_{r=R} = v_{re} \mid_{r=R} \tag{6}$$

and the normal component of the momentum flux,

$$p_i + \frac{B^2}{8\pi} \mid_{r=R} = p_e \mid_{r=R} \tag{7}$$

should be satisfied. We choose the cylindrical coordinate system with the z-axis directed along magnetic field.

According the linear theory developed in [3] for perturbations, proportional to $\exp(-i\omega t + im\varphi + ikz)$, the MHD set is reduced to a single equation for function ψ inside flux tube

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \psi}{\partial r} + \left[\frac{\omega^2}{v_A^2} - k^2 - \frac{1}{r^2}\right]\psi = 0 \tag{8}$$

and to a single equation for the velocity potential $\mathrm{v} = -\nabla\chi$ outside it:

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \chi}{\partial r} + \left[\frac{\omega^2}{c_{se}^2} - k^2 - \frac{1}{r^2}\right]\chi = 0. \tag{9}$$

Here v_A is the Alfven velocity and $c_S = \sqrt{\frac{\gamma p}{\rho}}$ is the sound speed outside flux tube. The velocity and magnetic field perturbations are expressed through the function ψ as follows:

$$v_r = -\frac{\partial \psi}{\partial r}, \quad v_\varphi = -\frac{1}{r}\frac{\partial \psi}{\partial \varphi}, \quad v_z = 0, \tag{10}$$

$$b_r = \frac{kB}{\omega}\frac{\partial \psi}{\partial r}, \quad b_\varphi = \frac{im}{r}\frac{kB}{\omega}\psi, \quad b_z = -\frac{i\omega B}{c_{si}^2 + v_A^2}\left[1 - \frac{k^2 v_A^2}{\omega^2}\right]\psi. \tag{11}$$

Respectively, the pressure perturbation outside flux tube is as

$$\delta p_e = -i(\omega - ku)\rho_e \chi. \tag{12}$$

Inside the flux tube the solution is proportional to the first order ($m = 1$) Bessel function: $\psi = A\mathcal{J}_1(q_i r)$ with $q_i^2 = \frac{\omega^2 - k^2 v_A^2}{v_A^2}$. In outer region the solution of (9) should have a form of a divergent wave: $\chi = D\mathcal{H}_1^{(1)}(q_e r)$, where $\mathcal{H}$ is a Hankel function and $q_e^2 = \frac{\omega^2 - k^2 c_{se}^2}{c_{se}^2}$.

The linearized boundary conditions (6), (7) lead to the following dispersion relation:

$$\eta \frac{\omega^2 - k^2 v_A^2}{\omega^2} \frac{\partial \ln \chi}{\partial r} = \frac{\partial \ln \psi}{\partial r} \qquad (13)$$

where we use the denotation $\eta = \frac{\rho_i}{\rho_e}$. In the long wavelength limit (2) the expression (13) is a subject of expansion in powers of a small parameter (kR). The first term in this expansion gives the phase velocity of a kink mode (1). Retaining the next order terms we obtain the following dispersion relation:

$$\omega = ck + \beta k^3 + i\mu_k^3 \qquad (14)$$

where

$$\beta = -\frac{cR^2}{8(1+\eta)^2}, \quad \mu = -\frac{\pi c R^2}{4} \frac{c^2 - c_s^2}{(1+\eta)c_s^2}. \qquad (15)$$

Here the second term describes a weak dispersion of a kink mode, and the third term corresponds to the effect of radiative damping of flux tube oscillations described in [3]: according this effect the oscillating flux tube gives off its energy through the radiation of a secondary acoustic waves.

To find out the character of nonlinearity of a kink mode note, that the first nonlinear term that can have an effect on the finite amplitude kink oscillations is a cubic one. As the azimuthal dependence of the quadratic nonlinearity contains only the terms with $m = 0$ and $m = 2$. Taking this fact into account and being consistent with the dispersion relation (14), we introduce the following stretched variables:

$$\zeta = \epsilon(z - ct), \quad \tau = \epsilon^2 t. \qquad (16)$$

To carry out the adequate perturbation expansion of the MHD equations we represent the velocity and magnetic field having power series expansion in ϵ as follows:

$$\begin{aligned} \mathbf{v}_\perp &= \epsilon^{1/2}\mathbf{v}_{1\perp} + \epsilon^{3/2}\mathbf{v}_{2\perp} + \cdots, \\ v_z &= \epsilon v_{1z} + \epsilon^2 v_{2z} + \cdots, \\ \mathbf{B}_\perp &= \epsilon^{1/2}\mathbf{B}_{1\perp} + \epsilon^{3/2}\vec{B}_{2\perp} + \cdots, \\ B_z &= B_0 + \epsilon^{3/2} B_{1z} + \epsilon^{5/2} B_{2z} + \cdots, \\ \rho &= \rho_0 + \epsilon\rho_1 + \epsilon^2\rho_2 + \cdots. \end{aligned} \qquad (17)$$

Outside flux tube we have, respectively,

$$\begin{aligned} \mathbf{v}_{e\perp} &= \epsilon^{1/2}\mathbf{v}_{e1\perp} + \epsilon^{3/2}\mathbf{v}_{e2\perp} + \cdots, \\ v_{ez} &= \epsilon^{3/2} v_{e1z} + \epsilon^{5/2} v_{e2z} + \cdots, \\ p_e &= p_{e0} + \epsilon^{3/2} p_{e1} + \epsilon^{5/2} p_{e2} + \cdots, \\ \rho_e &= \rho_{e0} + \epsilon^{3/2}\rho_{e1} = \epsilon^{5/2}\rho_{e2} + \ldots \end{aligned} \qquad (18)$$

where $\mathbf{v}_\perp$ and $\mathbf{B}_\perp$ are the transverse (r and φ) components of velocity and magnetic field. The expansions (17) and (18) give the correct description of a linear stage of flux tube oscillations and are consistent with the main features of weakly dispersive long wavelength transverse oscillations of flux tube embedded in nonmagnetic region. It is important to note that in this limit the above choice allows to specify the character of nonlinearity separately from the (weak) dispersion which, in this limit, has a form obtained from linear analyze. Substituting (17) and (18) in the MHD set, and equiting terms of each order in ϵ, we obtain a sequence of equations up to the desired order. First at the order of $\epsilon^{3/2}$ we have from (3), (4):

$$-c\rho_0 \frac{\partial \mathbf{v}_{1\perp}}{\partial \zeta} = -\frac{B_0}{4\pi}\nabla_\perp B_z + \frac{B_0}{4\pi}\frac{\partial \mathbf{B}_{1\perp}}{\partial \zeta}, \tag{19}$$

$$-c\frac{\partial \mathbf{B}_{1\perp}}{\partial \zeta} = B_0 \frac{\partial \mathbf{v}_{1\perp}}{\partial \zeta} \tag{20}$$

and for outer region:

$$-c\rho_{e0}\frac{\partial \mathbf{v}_{e1\perp}}{\partial \zeta} = -\nabla_\perp p_1, \tag{21}$$

$$p_1 = c_s^2 \rho_1 \tag{22}$$

From (20) we have

$$\mathbf{B}_{1\perp} = -\frac{B_0}{c}\mathbf{v}_{1\perp}. \tag{23}$$

Substituting $\mathbf{B}_{1\perp}$ from (23) into (19) we have

$$\left(-c\rho_0 + \frac{B_0^2}{4\pi c}\right)\frac{\partial \mathbf{v}_{1\perp}}{\partial \zeta} = -\nabla_\perp \frac{B_0 B_{1z}}{4\pi}. \tag{24}$$

One can see from (24) that $\mathbf{v}_{1\perp}$, and respectively, $\mathbf{B}_{1\perp}$, can be expressed through some function ψ exactly in the same way as above (cf. Eqs. (10), (11)). That is:

$$\mathbf{v}_{1\perp} = -\nabla_\perp \psi, \quad \mathbf{B}_{1\perp} = \frac{B_0}{c}\nabla_\perp \psi, \quad B_{1z} = \frac{B_0}{c}\frac{v_A^2 - c^2}{v_A^2}\frac{\partial \psi}{\partial \zeta}. \tag{25}$$

Using the boundary conditions (6), (7) yelds:

$$\left(-c\rho_0 + \frac{B_0^2}{4\pi c}\right)\frac{\partial \mathbf{v}_{1\perp}}{\partial \zeta} = -c\rho_{e0}\frac{\partial \mathbf{v}_{1\perp}}{\partial \zeta}. \tag{26}$$

This expression coincides with those obtained in [3] for the linear oscillations of flux tube and gives the phase velocity as (1). The validity of (26) can be readily shown by the integrating of the r-component of Eq.(24) over r in a whole space. Indeed, lets us represent ψ and χ functions as follows:

$$\psi = A X_i(r) e^{i\varphi}\xi_i(z,t), \quad \chi = D X_e(r) e^{i\varphi}\xi_e(z,t). \tag{27}$$

Integrating the r-component of Eq. (24) in a whole space with the help of (27) we have

$$\int_0^R \left(-c\rho_0 + \frac{B_0^2}{4\pi c}\right) A \frac{\partial X_i(r)}{\partial r}\frac{\partial \xi_i}{\partial \zeta} dr - \int_R^\infty c\rho_e D \frac{\partial X_e}{\partial r}\frac{\partial \xi_e}{\partial \zeta} dr = \int_0^R \frac{\partial}{\partial r}\frac{B_0 B_{1z}}{4\pi} dr + \int_R^\infty \frac{\partial p_{1e}}{\partial r} dr. \tag{28}$$

Continuity of the momentum flux eliminates r.h.s. of Eq. (28), and a continuity of the normal component of velocity leads streightforward to Eq. (26).

Next, at the order of ϵ^2 we have from the second equation of (4) and from the z-component of (3) we have, respectively,

$$\rho_1 = \frac{\rho_0}{c} v_{1z}, \quad \rho_0 c v_{1z} = \frac{c^2}{v_A^2} \frac{B_{1\perp}^2}{8\pi}. \tag{29}$$

Here we used relationships (25), and their consequence in a form:

$$\nabla_\perp B_{1z} = \frac{v_A^2 - c^2}{v_A^2} \frac{\partial \mathbf{B}_{1\perp}}{\partial \zeta}. \tag{30}$$

Now, at the order of $\epsilon^{5/2}$ transverse components of equations (3) and the first equation of (4) give:

$$-c\rho_0 \frac{\partial v_{2\perp}}{\partial \zeta} - \frac{B_0}{4\pi} \frac{B_{2\perp}}{\partial \zeta} + \frac{B_0}{4\pi} \nabla_\perp B_{2z} = -\rho_0 \frac{\partial v_{1\perp}}{\partial \tau} + c\rho_1 \frac{\partial v_{1\perp}}{\partial \zeta} - \rho_0 v_{1z} \frac{\partial v_{1\perp}}{\partial \zeta} \tag{31}$$

and,

$$c \frac{B_{2\perp}}{\partial \zeta} + B_0 \frac{\partial v_{2\perp}}{\partial \zeta} = \frac{\partial B_{1\perp}}{\partial \tau} + \frac{\partial}{\partial \zeta}(v_{1z} B_{1\perp}). \tag{32}$$

In outer region at this order we have:

$$-c\rho_{e0} \frac{\partial \mathbf{v}_{e2\perp}}{\partial \zeta} + \nabla_\perp p_2 = -\rho_{e0} \frac{\partial \mathbf{v}_{e1\perp}}{\partial \tau} - \frac{\rho_{e1}}{2} \nabla_\perp \mathbf{v}_{e1}^2, \tag{33}$$

$$-c\rho_{e0} \frac{\partial v_{e1z}}{\partial \zeta} = -\frac{\partial p_1}{\partial \zeta} \tag{34}$$

or, taking into account that $p_1 = c_s^2 \rho_{e1}$, instead of (34) we have

$$c\rho_{e0} v_{e1z} = \rho_{e1} c_s^2. \tag{35}$$

Combining equations (33) and (35), and taking into account that

$$\nabla \rho_{e1} = \frac{c\rho_{e0}}{c_s^2} \frac{\partial \mathbf{v}_{e1\perp}}{\partial \zeta} \tag{36}$$

we can rewrite Eq. (33) as follows:

$$-c\rho_{e0} \frac{\partial \mathbf{v}_{e2\perp}}{\partial \zeta} + \nabla_\perp \left(p_2 + \frac{\rho_{e1} \mathbf{v}_{e1\perp}^2}{2} \right) = -\rho_{e0} \frac{\partial \mathbf{v}_{e1\perp}}{\partial \tau} + \frac{c\rho_{e0}}{c_s^2} \mathbf{v}_{e1\perp}^2 \frac{\partial \mathbf{v}_{e1\perp}}{\partial \zeta}. \tag{37}$$

Matching now the equations (31) and (37) through the boundary conditions we obtain:

$$c(\rho_{i0} + \rho_{e0}) \frac{\partial \mathbf{v}_{2\perp}}{\partial \zeta} + \frac{B_0}{4\pi} \frac{\partial \mathbf{B}_{2\perp}}{\partial \zeta} = c(\rho_{i0} + \rho_{e0}) \frac{\partial \mathbf{v}_{1\perp}}{\partial \tau} - c \frac{\rho_{e0}}{2c_s^2} \mathbf{v}_{1\perp}^2 \frac{\partial \mathbf{v}_{1\perp}}{\partial \zeta}. \tag{38}$$

Eliminating a second order terms from Eqs. (38) and (32) we obtain streightforward nonlinear equations with respect to stretched variables:

$$2 \frac{\partial \mathbf{B}_{1\perp}}{\partial \tau} + \frac{c}{\rho_{i0} v_A^2} \frac{\partial}{\partial \zeta} \left(\mathbf{B}_{1\perp} \frac{\mathbf{B}_{1\perp}^2}{8\pi} \right) - \frac{c^2}{c_s^2} \frac{c\rho_{e0}}{B_0^2(\rho_{i0} + \rho_{e0})} \frac{\mathbf{B}_{1\perp}^2}{2} \frac{\partial \mathbf{B}_{1\perp}}{\partial \zeta}. \tag{39}$$

It is convenient to introduce instead of the transverse components $\mathbf{B}_\perp$ the complex magnitude $H = B_r - iB_\varphi$ and normalize it by the unperturbed magnetic field B_0. Then, finally, the nonlinear equation for a kink mode has a form:

$$\frac{\partial H}{\partial \tau} + \frac{c}{4}\frac{\partial}{\partial \zeta}(|H|^2 H) - \frac{c^3}{4(1+\eta)c_s^2}|H|^2\frac{\partial H}{\partial \zeta} = 0. \tag{40}$$

The equation (40) apart the last term is similar to those obtained for hydromagnetic waves parallel to the magnetic field in a cold plasma [6,7]. We would like to emphasize that unlike the case of an unbounded plasma considered in [6] and [7] here we deal with oscillating magnetic string interacting with the nonmagnetic surroundings. The influence of a magnetic free region is provided by the last term of Eq. (40) as well as is reflected in the propagation speed of a kink mode (cf.(1)) which contains the plasma density outside flux tube. The equation (40) should be completed by the dispersion and radiative damping terms obtained in a linear analyze. With the help the dispersion relation (14) a standard procedure gives the equation describing the evolution of a weakly nonlinear and weakly dispersive oscillations of magnetic flux tube corresponding to dipole (kink) mode, and contains the condition when oscillating flux tube can radiate secondary acoustic waves:

$$\frac{\partial H}{\partial \tau} + \frac{c}{4}\frac{\partial}{\partial \zeta}(|H|^2 H) - \frac{c^3}{4(1+\eta)c_s^2}|H|^2\frac{\partial H}{\partial \zeta} + \beta\frac{\partial^3 H}{\partial \zeta^3} + \frac{\mu}{\pi}\text{v.p.}\int_{-\infty}^{\infty}\frac{\partial^3 H}{\partial \zeta^3}\frac{ds}{\zeta - s} = 0 \tag{41}$$

with β and μ given by expressions (15). The effect of radiative damping is very important when studying the dynamics of flux tube in a presence of shear mass flows along the magnetic field [4]. In this case as it was shown in [4] along magnetic flux tube negative energy waves (NEW) can be excited. According the main feature of NEW consisting in growing their amplitudes due to the any kind of dissipation, the radiative damping provides the development of strong instability in those region where NEW can be excited. In the next paper of a present series [8], we derive the evolution equation similar to those obtained above in a presence of shear mass flow along the magnetic flux tube, and show that depending on the parameters of flux tube and surrounding plasma nonlinear equation of the type of (41) with a source of energy as a shear flow leads to vigorous nonlinear dynamics of flux tube, such as appearance of a solitons with explosively growing amplitudes.

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