Письма в ЖЭТФ, том 58, вып.8, стр.598 - 602 ©1993 г. 25 октября

GRAVITATIONAL DESCENDANTS AS GENERATORS OF DIFFEOMORPHISMS OF THE TARGET SPACE IN TOPOLOGICAL LANDAU - GINZBURG GRAVITY

*A.Losev, I. Polyubin**

Institute of Theoretical and Experimental Physics
117259 Moscow, Russia

**L.D.Landau Institute for Theoretical Physics RAS*
117334 Moscow, Russia

Submitted 2 September 1993

We study flows on the space of topological Landau–Ginzburg theories coupled to topological gravity. We argue that flows corresponding to gravitational descendants change the target space from a complex plane to a punctured complex plane and lead to the motion of punctures.It is shown that the evolution of the topological theory due to these flows is given by dispersionless limit of KP hierarchy.

1.Introduction. One of the most interesting features of topological matter coupled to topological gravity [1–6] is its connection to integrable systems. Namely, the exponent of the generating function for correlators turns out to be tau-function of the integrable system. We argue that this connection can be explained as follows.

Observables can be identified with the tangent vectors to the space of topological theories,thus, each observable leads to the flow on the space of topological theories. The very existence of the generating function for the correlators means that these flows commute.

Here we show that flows corresponding to descendants change the target space of the theory: from a complex plane to a punctured complex plane. After that change flows correspond to motion of punctures. Integrable system appearing in the theory with one-dimensional target space with superpotential X^p is dispersionless limit of p-reduced KP hierarchy.

2. Landau–Ginzburg Topological Matter Theory on punctured plane. Topological Matter LG theory is a type B twisted N=2 supersymmetric sigma model [5, 8]. We consider as a target space of this model a punctured complex space with coordinate X on it. $N = 2$ model described by Kahler prepotential that we take as $|Q(X)|^2$ and superpotential W-polynomial of degree p. Kahler metric $|Q'(X)|^2$ has zeros at punctures. Lagrangian of the model (written in terms of twisted superfields) is:

$$L = \int d^4\theta |Q(X)|^2 + \int d^2\theta_+ W(X) + \int d^2\theta_- \bar{t}\bar{W}(\bar{X}) \tag{1}$$

This model if well-defined for all values of $\bar{t}$, and correlators in Topological matter are independent of $\bar{t}$, but in coupling to topological gravity the $\bar{t}$ dependence aprrears(holomorphic amomaly of Bershadsky, Cecotty, Ooguri and Vafa). All our considerations below assume that $\bar{t}$ tends to zero.

The space of local topological observables is given by functions, that are holomorphic everywhere except punctures. We take them in the form $P(X)/Q'(X)$, where P is a polynomial in X.

The n-point correlators in genus zero of the worldsheet in this topological matter theory are equal to:

$$< P_1/Q', \ldots, P_n/Q' >^M_{W,Q} = \int_\Gamma \frac{P_1(X)/Q' \ldots P_n(X)/Q'(dQ)^2}{dW}. \tag{2}$$

Here the contour of integration Γ separates zeros of dW from infinity and punctures. Superscript M indicates that this is n-point correlator before coupling to topological gravity. (The three point correlator formula already appeared in [9]).

Note, that $(dQ)^2$ in the numerator is due to diffeomorphism anomaly in type B topological theories [10].

Formula (2) is true without any additional assumptions if $degW' > degQ'$, otherwise, the infinite point of the complex plane gives non-zero contribution. However, if we represent Q' as a perturbation of a constant and consider the n-point function as a seria in peturbation theory, then *perturbatively* the infinite point does not contribute to the answer. We will come to this subtle question in a separate publication.

3. Gravitational descendants. Gravitational descendants $\sigma_n(P/Q')$ can be constructed from matter fields using the same mechanism as in [11], (see also [12]) and are equal to:

$$\sigma_n^{W,Q}(P/Q') = (dW/dQ) \int \sigma_{n-1}(P/Q')dQ \tag{3}$$

here $\sigma_0(\Phi) = \Phi$.

4. Multipoint correlators in topological theory coupled to topological gravity in genus zero of the worldsheet. These n-point correlators $< \Phi_1, \ldots, \Phi_n >_{W,Q}$ are integrals over the moduli space of the complex structures of the worldsheet(sphere with n marked points). For $n = 3$ they are equal to correlators in the topological matter itself (2) (since in this case moduli space is given by one point). For $n > 3$ it is possible to write down recursion relations for these correlators,expressing n-point correlators in terms of $(n-1)$ point correlators. The recursion relations arise from integration over the position of one of the marked points on the worldsheet. This integration results in infitesimal shift of the background plus contact terms. In our case background consists of the pair W, Q; thus, we expect that fields will decompose in two sets: shifting W and shifting Q. Naimely, we uniquely decompose P/Q' as

$$P/Q' = S + RW'/Q', \tag{4}$$

where degree of S is less than degree of W. Then according to [4] S-term can be interpreted as a shift of superpotential, and , as we will see below, R-term will correspond to diffeomorphism, changing Q.

5. S-recursion relation. S-recursion relations have the following form: for $n > 2$

$$< P_1/Q', \ldots, P_n/Q', S >_{W,Q} = \frac{d}{dt} < P_1/Q', \ldots, P_n/Q' >_{W+tS,Q} +$$

$$+ \sum_{i=1}^{n} < P_1/Q', \ldots, C_{W,Q}(P_i/Q', S), \ldots, P_n/Q' >_{W,Q} . \tag{5}$$

One can argue [3, 11, 13] that contact term is given by the following representation:

$$SP_1/Q' = \Phi/Q' + (W'/Q') \int C_{W,Q}(P_1/Q', S) dQ. \tag{6}$$

This formula means, that the product of two fields (moving and standing at the marked point on the worldsheet) contains the first decsendant of the contact term. The arbitrariness is in the choice of the field Φ. This arbitrariness can be fixed by self-consistency requirement, and it turns out that in representation (6) Φ should have degree less than W. This contact term can be easily calculated:

$$C_{W,Q}(P_1/Q', S) = (1/Q')C_{W,X}(P_1, S) = (1/Q')(P_1 S/W')'_+. \tag{7}$$

Here subscript stands for the positive part in X expansion.

6. R-recursion relations R-recursión relations, that correspond to the integration over the position of the field RW'/Q' arise from the Ward identities connected with the diffeomorphism:

$$X \to X + tR(X)/Q'(X). \tag{8}$$

Due to this diffeomorphism in correlator $< P_1/Q', \dots, P_n/Q' >_{W,Q}$ its ingredient vary as follows:

$$\begin{array}{rcl} W & \to & W + tW'R/Q', \\ Q & \to & Q + tR, \\ P_i/Q' & \to & P_i/Q' + t(P_i/Q')'R/Q' = P_i/Q' - tC^{cl}_{W,Q}(P_i/Q', RW'/Q'). \end{array} \tag{9}$$

The variation of superpotential in the action leads to the integral over the worldsheet that simulates integral over position of the insertion of $W'R/Q'$. The change of P_i due to this diffeomorphism will be interpreted below as minus the "classical" contact term C^{cl}. But we know that there is also a "topological gravity" contact term C^{tg}, see (6):

$$P_i RW'/(Q')^2 = (W'/Q') \int C^{tg}_{W,Q}(P_i/Q', RW'/Q') dQ \tag{10}$$

thus

$$C^{tg}_{W,Q}(P_i/Q', RW'/Q') = (P_i R/Q')'/Q'. \tag{11}$$

Now we are ready to use Ward identities; they state (for $n > 2$)

$$< P_1/Q', \dots, P_n/Q', RW'/Q' >_{W,Q} = \frac{d}{dt}(< P_1/Q', \dots, P_n/Q' >_{W,Q-tR})\Big|_{t=0} +$$

$$+ \sum_{i=1}^{n} < P_1/Q', \dots, C^{tot}_{W,Q}(P_i/Q', RW'/Q'), \dots, P_n/Q' >_{W,Q}, \tag{12}$$

where the total contact term is given by:

$$C^{tot}_{W,Q}(P_i/Q', RW'/Q') = C^{tg}_{W,Q}(P_i/Q', RW'/Q') + C^{cl}_{W,Q}(P_i/Q', RW'/Q') = P_i R'/(Q')^2. \tag{13}$$

Thus, the R-recursion relation takes the following simple form:

$$< P_1/Q', \ldots, P_n/Q', RW'/Q' >_{W,Q} = \frac{d}{dt} < P_1/(Q-tR)', \ldots, P_n/(Q-tR)' >_{W,Q-tR} \Big|_{t=0}, \tag{14}$$

R and S recursion relations together with the expression (2) of 3-point correlator completely define n-point correlators.

7.Integrable system. In topological theory coupled to topological gravity the central object is the generating function of n-point correlators. Defining correlators in the presence of "formal exponent" as :

$$\begin{array}{rcl} < P_1, \ldots, P_n; \exp(P) >_W & = & < P_1, \ldots, P_n >_W + < P_1, \ldots, P_n, P >_W + \\ +\frac{1}{2} < P_1, \ldots, P_n, P, P >_W & + & \ldots + \frac{1}{k!} < P_1, \ldots, P_n, P, P, \ldots, P >_W + \ldots \end{array} \tag{15}$$

and using recursion relations (5),(14) one can show that:

$$< P_1/Q', \ldots, P_n/Q'; \exp(\sum_{k=1}^{\infty} t_k P_k/Q') >_{W,Q} = < P_1(t)/Q'(t), \ldots, P_n(t)/Q'(t) >_{W(t),Q(t)} \tag{16}$$

for some $W(t)$, $Q(t)$, $P_i(t)$ Moreover, if we define polynomials $R_j(t)$ and $S_j(t)$ as a R and S parts of the polynomial $P_j(t)$:

$$P_j(X,t) = W'(X,t)R_j(t) + Q'(X,t)S_j(t) \tag{17}$$

then one can show that

$$\frac{\partial}{\partial t_j} P_i(t) = C_{W,X}(P_i(t), S_j(t)) = (P_i(t)S_j(t)/W'(t))'_+, \tag{18}$$

$$\frac{\partial}{\partial t_j} W(t) = S_j(t), \tag{19}$$

$$\frac{\partial}{\partial t_j} Q(t) = -R_j(t). \tag{20}$$

8. Relation with dispersionless reductions of KP. Let $P_j(0) = X^j$ Using system (18)-(20) one can show that

A) The result of evolution of P_j equals to the derivative of Hamiltonian of dispersionless KP:

$$P_j(t) = (W(t)^{j/p})'_+. \tag{21}$$

B) The pair of functions (W,Q) evolute due to dispersionless KP: if we introduce a Poisson bracket between functions of t_1 and X as follows

$$\{T_1(X,t_1), T_2(X,t_1)\}_1 = \partial_1 T_1(X,t_1)\partial_X T_2(X,t_1) - \partial_1 T_2(X,t_1)\partial_X T_1(X,t_1) \tag{22}$$

then

$$\{W,Q\}_1 = 1 \tag{23}$$

and

$$\{W(X,t), (W(X,t)^{j/p})_+\}_1 = \partial_j W(X,t), \tag{24}$$

$$\{Q(X,t), (W(X,t)^{j/p})_+\}_1 = \partial_j Q(X,t). \tag{25}$$

C) The following observable

$$D=[\frac{p}{p+1}(W(t)^{p+1/p})'_+ - \sum_{j=1}^{\infty} t_j(W(t)^{j/p})'_+]/Q', \tag{26}$$

satisfies dilaton equation

$$< D, P_1/Q', \ldots, P_n/Q' >_{W,Q} = (n-2) < P_1/Q', \ldots, P_n/Q' >_{W,Q} . \tag{27}$$

Comparing this object with the dilaton that is a descendant of a puncture , we get Krichever form of the string equation in genus zero [9]:

$$\frac{p}{p+1}(W(t)^{p+1/p})'_+ - \sum_{j=1}^{\infty} t_j(W(t)^{j/p})'_+ = QW' \tag{28}$$

that algebraically determines W and Q in terms of times t.

D) Finally, $<;\exp(\sum_k kt_k X^{k-1}) >_{W,X}$ is a logarithm of tau-function of dispersionless W-reduced KP hierarchy.

9. Conclusions. Approach used in this letter should be generalized to higher dimensions of the target space and to different topologies of the target space (for some steps in this direction see [14, 9]). Moreover, from the physical point of view, it should be generalized to the higher genera of the worldsheet (in A_n singularities see [15]). Development in this direction seems to link together such different theories as Singularity theory, Geometry of moduli space of Riemann surfaces and 2-dimensional quantum gravity.

We are deeply indebted to R.Dijkgraaf,A.Gerasimov, V.Fock, I.Krichever, S.Kharchev, A.Marshakov, A.Mironov, A.Morozov, N.Nekrasov, A.Rosly, K.Saito, A.Varchenko for fruitful discussions.

---

1. E.Witten, Nucl. Phys. **B340**, 281 (1990).
2. R.Dijkgraaf and E.Witten, Nucl. Phys. **B342**, 486 ().
3. E.Verlinde and H.Verlinde, Nucl. Phys. **B348**, 457 (1991).
4. R.Dijkgraaf, E.Verlinde, and H.Verlinde, Proc. of the Trieste Spring School 1990, M.Green et al. eds., (World Scientific, 1991).
5. K.Li, Nucl. Phys. **354B**, 711 (1991).
6. M.Kontsevich, Comm. Math. Phys. **147**, 1 (1992).
7. C.Vafa, Mod. Phys. Lett.**A6**, 337 (1990).
8. T.Eguchi and S.K.Yang, Mod. Phys. Lett. **A5**, 1663 (1990).
9. I.M.Krichever, preprint LPTENS-92/18 (hepth 9205110).
10. E.Witten, Mirror Manifolds And Topological Field Theory, in S.-T.Yau, ed., Essays On Mirror Manifolds (International Press, 1992).
11. A.Losev, Descendants Constructed From Matter Fields And K. Saito Higher Residue Pairing In Topological L-G Theories Coupled To Topological Gravity, preprint TPI-MINN-40T (May,1992); A.Losev, TMP(Teoreticheskaya i matematicheskaya Fizika,in English) **95**, n.2, 307 (1993).
12. T.Eguchi et al, Topological Strings, Flat Coordinates And Gravitational Descendants, University of Tokyo preprint UT 630, Febr.1993.
13. R.Dijkgraaf, talk on RIMS conference "Infinite dimensional analisis", July, 1993.
14. B.Dubrovin, Nucl. Phys. **379B**, 627 (1992).
15. S.Kharchev et.al., Nucl. Phys., **380B**, (1992) 181 (1992).