Письма в ЖЭТФ, том 58, вып.8, стр.636 - 639 ©1993 г. 25 октября

DISTRIBUTION OF EXPONENTIAL DECAY RATES OF LOCALIZED EIGENFUNCTIONS IN FINITE QUASI 1*D* DISORDERED SYSTEMS

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Submitted 7 September 1993

For a quasi-1D disordered system of finite length L (a thick wire or a random banded matrix) we found the explicit form of the distribution of the quantity $r = |\psi(0)\psi(L)|^2$, with $\psi(0)$ and $\psi(L)$ being the values of an eigenfunction at the opposite ends of the sample. For a long sample the quantity $-\ln r$ is shown to have a normal distribution with a variance being twice as large as the mean. The latter determines the smallest Lyapunov exponent (inverse localization length).

It has been known for a long time that the logarithm of the resistance of a long 1D chain with a white-noise random potential is distributed asymptotically according the Gaussian law [1]. Even for a chain of finite length L it has turned out to be possible to derive an explicit analytical expression for the resistance distribution [2]. In contrast, the more realistic case of the disordered wire (i.e., a quasi-1D system with a large number of transverse channels $\Lambda \propto k_F^2 S$, S being the cross section of the wire and k_F being the Fermi wavenumber) appeared to be much harder to study analytically. The case of an asymptotically long wire was addressed in the papers[3, 4]. It was shown that the average density–density correlation function (and consequently the conductance) decreases exponentially; a corresponding decay rate was calculated [3]. This decay length ξ was found to be parametrically large $\xi \propto \Lambda l \gg l$ as compared to the mean free path l, in contrast to the strictly 1D case, where the condition $\xi \sim l$ always holds, and therefore there is no room for the diffusive regime $l \ll L \ll \xi$. However, the density–density correlation function (in contrast to its logarithm) is not a self-averaging quantity, so that the mean value gives no relevant information, and the whole distribution function is to be studied. This question was considered in [4], where the distribution of conductance of a long wire was proven to have asymptotically a log-normal form. However, parameters of this distribution were not explicitly calculated. An intensive numerical work performed in this direction (see recent review [5]) confirmed the normal form of the distribution of logarithm of conductance, and the variance/mean ratio was found to be the same as for the strictly 1D system with a white noise potential, i.e., equal to 2. In the opposite case of a small particle (that is equivalent to a wire with $L \ll \xi$) the conductance distribution was calculated in [6] for a case of weakly coupled point-like contacts. As to the case of the system of an arbitrary length, only the first two moments of the conductance distribution were calculated very recently [7] for the case of perfectly coupled leads.

Recently, interest in the statistical properties of quasi-1D systems was strongly stimulated by progress in the domain of "quantum chaos." In particular, it was discovered that quite generally the chaotic classical diffusion in the phase space can be suppressed by quantum interference effects. The most popular system for the investigation of this phenomenon, known as the "dynamical localization," is the so-called quantum kicked rotator (QKR) model [8]. The evolution operator $\hat{U}$ for the QKR that determines the dynamics of the system appears to have, in an appropriate basis, the structure of a (pseudo) "random banded matrix" (RBM) whose nonzero elements are concentrated in a band of width $b \gg 1$ around the main diagonal. This observation gave a boost to a numerical study of such matrices (see references in [9,10]).

Quite soon it was realized that all statistical properties of the RBM can be related to those of the thick wire in a simple way [9]. On a formal level the RBM model can be mapped onto the same 1D nonlinear supermatrix $\sigma-$model that was introduced in [3] for the disordered wire. Further study in the framework of this model revealed that it is possible to extract some important physical information for a system (a wire or RBM) of finite size; the distributions of the eigenfunction components, of the inverse participation ratios and of the local density of states were mentioned as particular examples [10].

In the present paper we are going to study the statistics of the following quantity:

$$r_\alpha(L) = |\psi_\alpha(0)\psi_\alpha(L)|^2, \tag{1}$$

where ψ_α stands for an eigenfunction corresponding to the energy level E_α and satisfying the Schrödinger equation $\hat{H}\psi_\alpha = E_\alpha\psi_\alpha$. The Hamiltonian $\hat{H}$ can be thought of as that of a thick disordered isolated wire; indices 0 (L) correspond to an arbitrary point situated at a distance shorter than the mean free path from the left (right) end of the sample. Equivalently, we can start from the 1D tight-binding model with L sites and the Hamiltonian belonging to the class of RBMs with a wide band, $b \gg 1$, and a matrix size $L \gg b$. In this case 0 (L) just means a site situated at a distance of the order of b or less from the left (right) end of the system.

The quantity r_α contains the essential information about the spatial structure of the eigenfunctions. In particular, $L^{-1}\ln r(L)$ determines at $L \gg \xi$ the rate of exponential decay of localized eigenfunctions (the smallest LE). The related quantity $f = \sum_\alpha r_\alpha$ determines probability for a quantum particle placed initially at the left edge to be found after infinite time at the right edge. A quantity of this sort was studied in [11] in the context of the dynamical localization.

The moments $r_k = \overline{\langle r_\alpha^k(L)\rangle_E}$, $k = 1, 2, \ldots$ (the bar and the angular brackets stand for, respectively, the disorder averaging and spectral averaging in a narrow energy window around the energy E), can be expressed in terms of Green functions $G(x, x', E) = (E - \hat{H})^{-1}_{xx'}$ as follows:

$$r_k = \frac{(k-1)!^2}{(2k-2)!}(2\pi\rho L)^{-1} \lim_{\eta\to 0)} (2\eta)^{2k-1}\overline{G^k(0,0,E+i\eta)G^k(L,L,E-i\eta)}, \tag{2}$$

with ρ being the 1D density of states. In order to calculate the disorder average [1] we employ a supersymmetric method [3,10]. In contrast to the strictly 1D

[1] Correlators similar to that in Eq. (2) were recently calculated in [12] for the strictly 1D case and a white noise potential. The author, however, did not use them for the purpose we do.

case the statistical properties of the disordered wire are dependent on the presence or absence of time-reversal symmetry (TRS). The case of broken TRS physically corresponds to a wire subjected to a strong magnetic field. The Hamiltonian of the system in this case can be represented by a Hermitean random matrix, whereas when the TRS is unbroken it belongs to the class of symmetric random matrices. Performing the calculation we find correspondingly

$$r_k = \frac{1}{4\pi^2 LS}\frac{1}{(2\gamma S)^{2k-1}}\frac{A_k}{(2k-2)!}\int_0^\infty d\nu\nu\sinh\pi\nu \mid \Gamma(k-1/2+i\nu/2)\mid^4 \exp\{-(1+\nu^2)\frac{L}{8\gamma}\}, \tag{3}$$

where the coefficient is $A_k = k!^2$ for the case of broken TRS and $A_k = (2k-1)!!^2$ otherwise, and $\Gamma(x)$ stands for the gamma–function. The parameter γ entering this equation has the dimensionality of a length and contains all the microscopic information about the disorder. More precisely, $\gamma = s\pi\rho D/2$, where D is a classical diffusion constant, and $s = 1$ (2) for the case of unbroken (broken) TRS. It is readily seen from Eq. (3) that all moments r_k decay asymptotically like $r_k \propto \exp(-L/8\gamma)$, and so the localization length ξ is proportional to γ. Let us mention that for the RBM case we have $S = 1$ and the parameter $\gamma \propto b^2 \gg 1$ [9]. Equation (3) is valid provided $L \gg 1$ ($L \gg l$) for the RBM (thick wire) case, respectively, the relation between L and γ being arbitrary.

Having at our disposal all moments r_k we are able to reconstruct the whole probability distribution of the dimensionless quantity $m_\alpha = 4\gamma^2 S^2 r_\alpha$. After some manipulations we obtain

$$\mathcal{P}(m) = \frac{8}{x\pi^2}\int_0^\infty d\nu\nu\sinh\pi\nu e^{-(1+\nu^2)x/4}\int_0^\infty dt t^{-3}K_0(t)K_{i\nu}(t)R(m,t), \tag{4}$$

where $R(m,t) = 4t^{-1}K_0(4m^{1/2}/t)$ when TRS is broken and $R(m,t) = \pi^{-1}m^{-1/2}K_0(2m^{1/2}/t)$ otherwise. Here $K_\mu(t)$ stands for the Macdonald function, and we have introduced the scaled length of the system $x = L/2\gamma$.

Expression (4) constitutes the main result of the present paper and describes the probability distribution of the quantity r_α for any relation between the system size L and the localization length $\xi \propto \gamma$.

For the case of short samples (the diffusive regime), $\gamma^{-1} \ll x \ll 1$, one can extract from Eq. (4) the distribution for the quantity $g_\alpha = L^2 S^2 r_\alpha \equiv x^2 m_\alpha$:

$$\mathcal{P}(g) = \begin{cases} 2K_0(2\sqrt{g}) \;, & \text{broken TRS,} \\ \dfrac{1}{\pi\sqrt{g}}K_0(\sqrt{g}) \;, & \text{unbroken TRS.} \end{cases} \tag{5}$$

These expressions have a transparent physical meaning: the values of the eigenfunction at points separated by a distance exceeding the mean free path are independent random variables whose distribution is equal to $\mathcal{P}(y) = \exp(-y)$ for broken TRS and $\mathcal{P}(y) = (2\pi y)^{-1/2}\exp(-y/2)$ for conserved TRS [6,10], y being equal to $LS|\psi|^2$, i.e., the same as the distribution of eigenfunction components in Gaussian ensembles.

In the opposite case of a very long wire (or a very large RBM), $x \gg 1$, the typical values of m are exponentially small. In the domain $z = -\ln m \gg 1$, $x \gg 1$, $z/x \sim 1$ one can notice that the integral over t in Eq. (4) is dominated by the region $t \ll 1$. Knowing that $K_{i\nu}(t \ll 1) = \frac{1}{2}[\Gamma(i\nu)t^{-i\nu} + \Gamma(-i\nu)t^{i\nu}]$, one can calculate this integral, using $x \gg 1$ to perform the remaining integration by the steepest descent method. This results in the following expression:

$$\mathcal{P}(z) = C(z/x)\frac{1}{2\pi^{1/2}x^{1/2}}\exp\{-\frac{(z-x)^2}{4x}\}, \tag{6}$$

where $C(u) = u\Gamma^2(\frac{3-u}{2})/\Gamma(u)$ for broken TRS and $C(u) = \pi^{-1}u\Gamma^2(1-u/2)/\Gamma(u)$ in the opposite case. We see that this probability distribution has asymptotically the form of the Gaussian distribution with the mean value $\overline{z} = x$ and the variance $\overline{(z-\overline{z})^2} = 2x$. The rate of the decay of all positive moments is four times smaller than $\overline{z}$ [see Eq. (3)]; i.e., again, this ratio is the same as for strictly 1D model with the white noise potential, but is different from the corresponding ratio for the Lloyd model[13]. These results are in good correspondence with the numerical data for both long quasi-1D systems [5] and the QKR model [11].

One of us (Y.V.F.) is grateful to B. Chirikov, F. Izrailev and D. Shepelyansky for stimulating discussions and appreciates the kind hospitality extended to him during his visits to the Como University and Institut für Theorie der Kondensierten Materie, Karlsruhe. The financial support by the Feinberg Fellowship, the program PHIT-NEDO (Y.V.F), and the Alexander von Humboldt Foundation (A.D.M) is acknowledged with thanks.

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