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HIGHER HAMILTONIAN STRUCTURES (the sl_2 CASE)

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We apply the procedure of Magri and Weinstein to produce an infinity of compatible Poisson structures on a bihamiltonian manifold, to the case of the KdV phase space. The higher Gel'fand–Dikii structures thus obtained contain non local terms, which we express with the help of the r.h.s. of the KdV hierarchy. We also give a generating function for all these Poisson structures, in terms of the Baker–Akhiezer functions. Finally we describe the symplectic leaves of these Poisson structures.

1. Definition of the higher Hamiltonian structures

In [M], [W], it is explained how to associate to a pair of Poisson structures V_i, $i = 1, 2$, on a manifold P, one of which (V_1) is symplectic, an infinite sequence V_n of compatible Poisson structures. The maps $V_i : T^*P \to TP$ allow to define the Nijenhuis operator $\Lambda : TP \to TP$ by $V_2 V_1^{-1}$; we then pose $V_n = \Lambda^{n-1} V_1$.

A natural example of compatible Poisson structures is provided by the first Gel'fand-Dikii and second (Adler-)Gel'fand-Dikii (GD1 and GD2) structures on manifolds of differential operators. Remark that GD1 is not exactly symplectic, since if we define the manifold to be $\mathcal{L}_2 = \{\partial^2 + u, u \in C_c^\infty(\mathrm{R})\}$, then $T_L\mathcal{L}_2 = \{\delta u, \delta u \in C_c^\infty(\mathrm{R})\}$ (the subscript c means compactly supported functions) for $L \in \mathcal{L}_2$, and $T_L^*\mathcal{L}_2$ is the set of forms given by $\delta u \mapsto \int_{\mathrm{R}} \xi \delta u$, $\xi \in C_c^\infty(\mathrm{R})$. The map V_1 is then $T_L^*\mathcal{L}_2 \to T_L\mathcal{L}_2$, $\xi \mapsto \xi'$; it has a cokernel of dimension one.

A natural way to overcome this difficulty is to enlarge the space of functionals on $\mathcal{L}_2$ (recall that in the usual formalism it is the set of maps $u \mapsto \int_{\mathrm{R}} f(x, u(x), \cdots, u^{(k)}(x))dx$, f smooth with compact support in x and polynomial in the other variables).

We now pose Fun $\mathcal{L}_2$ to be the linear span of the functionals

$$u \mapsto \int_{\mathrm{R}^k} f_1(x_1, u^{(\alpha)}(x_1)) \cdots f_k(x_k, u^{(\alpha)}(x_k)) \prod_{\alpha=1}^{N} \varepsilon(x_{i_\alpha} - x_{j_\alpha}) \prod_{i=1}^{k} dx_i \,,\; i_\alpha < j_\alpha \,,$$

with $\cup_{\alpha=1}^{k-1} \{i_\alpha, j_\alpha\} = \{1, \cdots, k\}$, and f_k polynomial in the $u^{(\alpha)}(x_k)$, $\alpha \geq 0$ with coefficients C_c^∞ functions of x_k ; ε is the Heaviside function ($\varepsilon(x) = -\frac{1}{2}$ if $x < 0$, $\frac{1}{2}$ else) ; N is arbitrary.

We still pose $T_L\mathcal{L}_2 = \{\delta u, \delta u \in C_c^\infty(\mathrm{R})\}$ but define now $T_L^*\mathcal{L}_2$ as the set of forms given by $\delta u \mapsto \int_{\mathrm{R}} \tilde{\xi} \delta u$, $\tilde{\xi}$ a smooth function on R with opposite limits at $+\infty$ and $-\infty$. The map $V_1 : T_L^*\mathcal{L}_2 \to T_L\mathcal{L}_2$, $\tilde{\xi} \mapsto \tilde{\xi}'$ is now a linear isomorphism and we can try to apply the Magri-Weinstein procedure.

Let us consider in $T^*_L\mathcal{L}_2$ the subspace T^*_0 linearly spanned by the functions

$$x_1 \mapsto \int_{\mathbb{R}^{k-1}} f_1(x_1, u^{(\alpha)}(x_1)) \dots f_k(x_k, u^{(\alpha)}(x_k)) \prod_{\alpha=1}^{N} \varepsilon(x_{i_\alpha} - x_{j_\alpha}) \prod_{i=2}^{k} dx_i\,,$$

where exactly one i_α or j_α is equal to one and $f_1, \dots, f_k$ are as above, but the constant term of f_1 is of the form constant $+C^\infty_c$ (N is arbitrary); and in $T_L\mathcal{L}_2$ the subspace T_0 linearly spanned by analogous expressions, with the conditions that the constant terms of the polynomial f_i are C^∞_c (and the other are smooth), and no restriction on (i_α), (j_α).

The map V_1 induces a linear isomorphism between T^*_0 and T_0 : indeed, $V_1(T^*_0) \subset T_0$, and the preimage of the element of T_0

$$x_1 \mapsto \int_{\mathbb{R}^{k-1}} f_1(x_1, u^{(\alpha)}(x_1)) \dots f_k(x_k, u^{(\alpha)}(x_k)) \prod_{\alpha=1}^{N} \varepsilon(x_{i_\alpha} - x_{j_\alpha}) \prod_{i=2}^{k} dx_i\,,$$

is

$$x_0 \mapsto \int_{\mathbb{R}^k} 1(x_0) f_1(x_1, u^{\alpha}(x_1)) \dots f_k(x_k, u^{(\alpha)}(x_k)) \cdot \varepsilon(x - x_1) - \prod_{\alpha=1}^{N} \varepsilon(x_{i_\alpha} - x_{j_\alpha}) \prod_{i=1}^{k} dx_i$$

which belongs to T^*_0 ($1(x_0)$ is the constant function of x_0 equal to 1). On the other hand, $V_2(T^*_0) \subset T_0$ (recall that $V_2 = \frac{1}{4}(\partial^3 + 4u\partial + 2u_x)$). Note also that V_1 and V_2 define antisymmetric bilinear forms on T^*_0 (or more generally on the space $T^*(-1)$, if $T^*(c) = \{\xi \in C^\infty(\mathbb{R}) | \xi$ is constant at both infinities and $\lim_{+\infty}\xi = c\lim_{-\infty}\xi\}$; if $\xi, \eta \in T^*(c)$, $\langle V_1(\xi), \eta\rangle + \langle \xi, V_1(\eta)\rangle = (1-c^2)\lim_{+\infty}\xi\lim_{+\infty}\eta$; and V_1 is not bijective from $T^*(1)$ to $C^\infty_c(\mathbb{R})$. On the other hand, V_2 defines an antisymmetric form on any $T^*(c)$.) We can check that V_1 and V_2 still define Poisson structures on Fun $\mathcal{L}_2$.
The recursion operator Λ is then well defined from T_0 to itself ; we get mappings $V_n = \Lambda^n\partial$ from T^*_0 to T_0.

If now F belongs to Fun $\mathcal{L}_2$, dF belongs to T^*_0, and $V_n(dF)$ to T_0 ; let then G be an other function of Fun $\mathcal{L}_2$. It is easy to see that $V_n(dF)G$ still belongs to Fun $\mathcal{L}_2$. Thus, the V_n define bilinear operations on Fun $\mathcal{L}_n$. The Magri-Weinstein arguments allow to show that they form an infinite family of compatible Poisson brackets, that we will call higher Gel'fand–Dikii brackets. We will now analyse these brackets more closely.

2. Non local part of the higher Gel'fand–Dikii structures

Recall that on the KdV phase space $\mathcal{L}_2 = \{\partial^2 + u, u \in C^\infty_c(\mathbb{R})\}$, GD1 is given by the operator $V_1 = \partial$ and GD2 by $V_2 = \frac{1}{4}(\partial^3 + 2u\partial + 2u_x)$, leading to the brackets

$$\{u(x), u(y)\}_1 = \delta'(x-y),\ \{u(x), u(y)\}_2 = \frac{1}{4}\delta'''(x-y) + \frac{1}{2}(u(x) + u(y))\delta'(x-y)\,.$$

The recursion operator is $\Lambda = \frac{1}{4}(\partial^2 + 4u + 2u_x\partial^{-1})$. The operator of the n-th GD structure is then $\Lambda^n\partial$. The result of this section is that $\Lambda^n\partial$ can be written (local part) + $\sum_{i=1}^{n} K'_i\partial^{-1}K_{n-i}\partial$, where local part is a differential operator (with coefficients differential polynomials in u), and where K_n are the right hand sides of the KdV hierarchy : $K_{n+1} = \Lambda^* K_n$ (we set $(f\partial^k)^* = (-\partial)^k f$, f a function, k

an integer), $K_0 = 1$; K_n is then a polynomial differential in u, containing (by assumption) no constant term if $n \neq 0$.

We first prove some relations in the algebra of formal pseudodifferential operators :

Lemma.— *If a, b, a_1, b_1 are differential polynomials in u and P is a differential operator, we have*

$$P(a\partial^{-1}b) = [Pa]\partial^{-1}b + R\,,$$

$$(a\partial^{-1}b)P = a\partial^{-1}[P^*b] + S\,,$$

$$(a\partial^{-1}b)(a_1\partial^{-1}b_1) = [a\partial^{-1}ba_1]\partial^{-1}b_1 + a\partial^{-1}[(a_1\partial^{-1}b_1)^*b]$$

where e.g. $[Pa]$ denotes the function obtained from the operation of P on a, and R and S are differential operators (in the last relation, ba_1 as assumed to be a total derivative c' and $[\partial^{-1}ba_1] = c$ in the r.h.s.).

The first relation is proved by examining the cases P = function and $P = \partial$, and by the remark that the set of P satisfying it forms an algebra ; the second relation can be deduced from it by applying $*$. To obtain the third one, we replace in the r.h.s. ba_1 by $\partial[\partial^{-1}ba_1] - [\partial^{-1}ba_1]\partial$.

We apply this lemma to the computation of the non local part of Λ^n. $\Lambda =$ (local part) + $K_1'\partial^{-1}$ since $K_1 = \frac{u}{2}$. Assume that Λ^n is written $P_n + \sum_{i=1}^{n} K_i'\partial^{-1}K_{n-i}$, P_n a differential operator. Then

$$\Lambda^{n+1} = \text{differential operator} + \frac{1}{4}[P_n.2u_x]\partial^{-1} +$$

$$+\sum_{i=1}^{n} K_i'\partial^{-1}\left[\frac{1}{4}(\partial^2+4u)^*K_{n-i}\right] + \sum_{i=1}^{n} K_i'\left[\frac{\partial^{-1}}{4}K_{n-i}2u_x\right]\partial^{-1} -$$

$$-\frac{1}{4}\sum_{i=1}^{n} K_i'\partial^{-1}[\partial^{-1}2u_xK_{n-i}] =$$

$$= [\Lambda^n\frac{u_x}{2}]\partial^{-1} + \sum_{i=1}^{n} K_i'\partial^{-1}K_{n+1-i} + \text{diff. op.} =$$

$$= K_{n+1}'\partial^{-1} + \sum_{i=1}^{n} K_i'\partial^{-1}K_{n+1-i} + \text{diff. op.}$$

since from $\Lambda^* = \partial^{-1}\Lambda\partial$ follows $K_{n+1}' = \Lambda K_n'$ and so $K_{n+1}' = \Lambda^n K_1' = \Lambda^n\frac{u_x}{2}$. From $K_{s+1} = \Lambda^* K_s$ follows that $2u_xK_s$ is the total derivative of a differential polynomial in u; in $[\partial^{-1}2u_xK_s]$ we fix the constant term of this polynomial to zero.

This establishes Λ^n = local part + $\sum_{i=1}^{n} K_i'\partial^{-1}K_{n-i}$ by induction. Our result follows.

The Poisson bracket for GDn is then :

$$\{u(x), u(y)\}_n = (\text{local part}) + \sum_{i=1}^{n-1} K_i'(x)\partial_x^{-1}K_{n-1-i}(x)\partial_x\delta(x-y) =$$

$$= (\text{local part}) - \sum_{i=1}^{n-2} K_i'(x)K_{n-1-i}'(y)\varepsilon(x-y)\,,$$

$$\text{(local part)} = \frac{1}{4^{n-1}}\delta^{(2n-1)}(x-y) + \sum_{i=0}^{2n-2} \tilde{p}_i(u)(x)\delta^i(x-y),$$

p_i being differential polynomials in u.

3. **A generating function for the higher GD structures.** Let ψ_λ and ψ^*_λ be conjugated wave (Baker-Akhiezer) functions for $\partial^2 + u$. That is, $(\partial^2+u)\psi_\lambda = \lambda^2\psi_\lambda$, $(\partial^2+u)\psi^*_\lambda = \lambda^2\psi^*_\lambda$, $\psi_\lambda(x) = e^{\lambda x}(1+\sum_{i\geq 1} u_i(x)\lambda^{-i})$, $\psi^*_\lambda(x) = e^{-\lambda x}(1+\sum_{i\geq 1} u_i(x)(-\lambda)^{-i})$, $u_i(x) = 0$ for x large negative enough. Pose $R_+ = \psi^2_\lambda$, $R_0 = \psi_\lambda\psi^*_\lambda$, $R_- = \psi^{*2}_\lambda$. The R_i satisfy $\frac{1}{4}(\partial^3 + 4u\partial + 2u_x)R_i = \lambda^2\partial R_i$. Then we have the identity

$$\frac{\lambda^2}{\Lambda - \lambda^2} = \frac{1}{2}R'_+\partial^{-1}R_- + \frac{1}{2}R'_-\partial^{-1}R_+ - R'_0\partial^{-1}R_0. \qquad (*)$$

It follows from

$$\frac{1}{4}(\partial^2+4(u-\lambda^2)+2u_x\partial^{-1})\frac{1}{2}R'_+\partial^{-1}R_- = \frac{1}{2}R'_+(\frac{1}{4}\partial^2+u-\lambda^2)\partial^{-1}R_- + \frac{1}{8}(2R''_+ + R'''_+)\partial^{-1}R_- +$$

$$+\frac{u_x}{4}R_+\partial^{-1}R_- - \frac{u_x}{4}\partial^{-1}R_+R_- = \frac{1}{8}\partial R'_+R_- + \frac{1}{4}R''_+R_- - \frac{u_x}{4}\partial^{-1}R_+R_-;$$

after summation of the two other terms, the last expression gives zero. We then have

$$(\Lambda-\lambda^2)(\frac{1}{2}R'_+\partial^{-1}R_- + \frac{1}{2}R'_-\partial^{-1}R_+ - R'_0\partial^{-1}R_0) = \frac{1}{8}(R''_+R_- + R''_-R_+ - 2R''_0R_0) =$$

$$= \frac{1}{4}(R_0'^2 - R'_+R'_-) = \frac{1}{4}(\psi_\lambda\psi^{*\prime}_\lambda - \psi'_\lambda\psi^*_\lambda)^2 = \lambda^2$$

(the Wronskian of ψ_λ and ψ^*_λ is constant, and takes the value 2λ at $-\infty$).

From $(*)$ follows a formula generating all the $\Lambda^k\partial$:

$$-\sum_{k\geq 0}\frac{\Lambda^k\partial}{\lambda^{2k}} = -\frac{1}{2}R'_+\partial^{-1}R'_- - \frac{1}{2}R'_-\partial^{-1}R'_+ + R'_0\partial^{-1}R'_0$$

(we have used $R_+R_- = R_0^2$). Posing $R_\pm = e^{\pm 2\lambda x}\hat{R}_\pm$, we have $R'_\pm = e^{\pm 2\lambda x}(\hat{R}'_\pm \pm 2\lambda\hat{R}_\pm$, and so

$$-\sum_{k\geq 0}\frac{\Lambda^k\partial}{\lambda^{2k}} = -\frac{1}{2}(\hat{R}'_+ + 2\lambda\hat{R}_+)(\partial-2\lambda)^{-1}(\hat{R}'_- - 2\lambda\hat{R}_-) -$$

$$-\frac{1}{2}(\hat{R}'_- - 2\lambda\hat{R}_-)(\partial+2\lambda)^{-1}(\hat{R}'_+ + 2\lambda\hat{R}_+) + R'_0\partial^{-1}R'_0.$$

Here $(\partial \pm 2\lambda)^{-1}$ should be expanded as $\sum_{k\geq 0}\frac{\partial^k}{(\pm 2\lambda)^{k+1}}$. So the only non local contribution is the one of the terms in R_0, which enables to recover the results of the last section (since $R_0 = 1 + \sum_{n\geq 1}\frac{K_n}{\lambda^{2n}}$). Remark also that all the expressions involved in the expansion in λ are polynomial differentials in u since they are invariant under the transformations $\psi_\lambda(x) \mapsto c(\lambda)\psi_\lambda(x)$, $\psi^*_\lambda(x) \mapsto c(\lambda)^{-1}\psi^*_\lambda(x)$, $c(\lambda) \in \mathbb{C}[[\lambda^{-1}]]^*$.

If w is then a primitive of the variable u, we get

$$\{w(x), w(y)\}_\lambda = \sum_{n\geq 0} \frac{\{w(x), w(y)\}_n}{\lambda^{2n}} =$$

$$= (\frac{1}{2}R_+(x)R_-(y) + \frac{1}{2}R_-(x)R_+(y) - R_0(x)R_0(y))\epsilon(x-y)$$

It should be remarked that, in view

$$R_+(x)R_-(y) + R_-(x)R_+(y) - 2R_0(x)R_0(y) = (\psi_\lambda(x)\psi^*_\lambda(y) - \psi^*_\lambda(x)\psi_\lambda(y))^2,$$

$$\{w(x), w(y)\}_\lambda = (\psi_\lambda(x)\psi^*_\lambda(y) - \psi^*_\lambda(x)\psi_\lambda(y))^2\epsilon(x-y).$$

Here $e^{\pm 2\lambda(x-y)}\epsilon(x-y)$ should be expanded as $-\sum_{k\geq 0}\frac{\delta^{(k)}(x-y)}{(\pm 2\lambda)^{k+1}}$.

In the case of GD3, the bracket is :

$$\{u(x), u(y)\}_3 = \frac{1}{16}\delta^V(x-y) + \frac{1}{4}(u(x)+u(y))\delta'''(x-y)+$$

$$+\frac{1}{8}(u'(x)-u'(y))\delta''(x-y) + \frac{1}{2}(u(x)^2+u(y)^2)\delta'(x-y) - \frac{1}{4}u'(x)u'(y)\epsilon(x-y).$$

For this third GD structure the functional $\int_{\mathbb{R}} u(x)dx$ is a Hamiltonian for the KdV equation. It is not clear what the Hamiltonians for the KdV equations are in the higher GD structures (it should be non local quantities related to the solutions of $(\partial^2 + u + \lambda)\varphi = 0$ for λ small).

Remarks. 1. R_+ and R_- can also be used to give a generating function for "A-operators" for Λ:

$$\frac{\partial\Lambda}{\partial t} = \sum_{n\geq 0}\frac{1}{\lambda^n}\frac{\partial\Lambda}{\partial t_n} = [\Lambda, R'_+\partial^{-1}R_- - R'_-\partial^{-1}R_+],$$

t_i being the times of the KdV hierarchy.

2. In the last section, we used the fact that $u_x K_n$ is a total derivative. We can show more generally that for any i and j, $K'_i K_j$ is a total derivative. Recalling that $R_0 = 1 + \sum_{n\geq 1}\frac{K_n}{\lambda^{2n}}$, it is enough to prove that $R_0(\lambda, x)R'_0(\mu, x)$ is a total derivative. Indeed (noting $W(f,g) = fg' - f'g$),

$$(W(\psi_\lambda, \psi^*_\mu)W(\psi^*_\lambda, \psi_\mu))' = (\mu^2 - \lambda^2)(R_0(\lambda)R'_0(\mu) - R'_0(\lambda)R_0(\mu)),$$

so that a primitive of $R_0(\lambda)R'_0(\mu)$ is

$$\frac{1}{2}(\frac{1}{\mu^2-\lambda^2}W(\psi_\lambda, \psi^*_\mu)W(\psi^*_\lambda, \psi_\mu) + R_0(\lambda)R_0(\mu));$$

it consists of differential polynomials in u since it is invariant by the transformations $\psi_\lambda(x) \mapsto c(\lambda)\psi_\lambda(x)$, $\psi^*_\lambda(x) \mapsto c(\lambda)^{-1}\psi^*_\lambda(x)$, $\psi_\mu(x) \mapsto d(\lambda)\psi_\mu(x)$, $\psi^*_\mu(x) \mapsto d(\lambda)^{-1}\psi^*_\mu(x)$, $c(\lambda) \in \mathbb{C}[[\lambda^{-1}]]^*$, $d(\mu) \in \mathbb{C}[[\mu^{-1}]]^*$.

4. Symplectic leaves of the higher GD structures.

Let M be a finite dimensional manifold, endowed with a symplectic structure $V_0 : T^*M \to TM$ and a Poisson structure $V_1 : T^*M \to TM$, which we assume to be compatible. Let $\Lambda = V_1 V_0^{-1}$ be the recursion operator.

Let $P(\Lambda) = \prod_{i=1}^{n} (\Lambda - \lambda_i)^{k_i}$ be an arbitrary monic polynomial (the λ_i are all different, $n \geq 1$). Then the general form of the Poisson structures defined in [W] is $V_P = P(\Lambda)V_0$.

Let us analyse the symplectic leaves of V_P. They are the integral manifolds of the forms ξ such that $P(\Lambda)V_0\xi = \prod_{i=1}^{n} (\Lambda - \lambda_i)^{k_i} V_0\xi = 0$. Any ξ satisfying this relation is such that at any point x of M, $\xi(x)$ belongs to the sum $\sum_{i=1}^{n} D_i(x)$ of the subspaces $D_i(x)$ of T_x^*M consisting of the forms satisfying $(\Lambda - \lambda_i)^{k_i} V_0\xi = 0$. So the tangent vectors to the symplectic leaves of V_P are exactly the vectors tangent to the symplectic leaves of all the $V_{(\Lambda-\lambda_i)^{k_i}}$, and the symplectic leaves of V_P are the intersections of the symplectic leaves of the $V_{(\Lambda-\lambda_i)^{k_i}}$.

Let us apply this discussion to the structures on the KdV phase space discussed above. First we describe the symplectic leaves of $V_{\Lambda^k V_0}$ $(k \geq 1)$.
For this, we solve $\Lambda^k V_0\xi = 0$, ξ a covector at $\partial^2 + u$. Recall ([K]) that the Casimir functions of V_1 are the functions of tr $M(u)$ ($M(u)$ is the monodromy operator of $\partial^2 + u$), so that $\Lambda V_0\xi = 0$ means that ξ is proportional to $d\mathrm{tr}\, M(u)$. Let us solve $\Lambda^2 V_0\xi = 0$.
Differentiating $(\Lambda + \lambda)V_0 d\,\mathrm{tr}\, M(u + \lambda) = 0$, we get $V_0 d\,\mathrm{tr}\, M(u) + \Lambda V_0 d\partial_\lambda \mathrm{tr}\, M(u + \lambda)|_{\lambda=0} = 0$.
$\Lambda^2 V_0\xi = 0$ means that $\Lambda V_0\xi$ proportional to $d\,\mathrm{tr}\, M(u)$, i.e. $V_0\xi$ is a linear combination of $d\,\mathrm{tr}\, M(u+\lambda)$ and $d\partial_\lambda \mathrm{tr}\, M(u+\lambda)|_{\lambda=0}$. In the same way, we see that $\Lambda^k V_0\xi = 0$ has for solutions the linear combinations of $d\,\mathrm{tr}\, M(u)$, $d\partial_\lambda \mathrm{tr}\, M(u+\lambda)|_{\lambda=0}, \ldots, d\partial_{\lambda^{k-1}} \mathrm{tr}\, M(u+\lambda)|_{\lambda=0}$.
Hence the symplectic leaves of $\Lambda^k V_0$ are the manifolds

$$\mathrm{tr}\, M(u) = C_0, \ldots, \partial_\lambda^{k-1} \mathrm{tr}\, M(u+\lambda)|_{\lambda=0} = C_{k-1},$$

$C_0, \ldots, C_{k-1}$ are constants. In Miura coordinates $[\partial^2 + u = (\partial + \varphi')(\partial - \varphi'), \varphi(-\infty) = 0]$ these conditions can be written

$$(e^\varphi + e^{-\varphi})(+\infty) = C_0,$$

$$e^{\varphi(\infty)} \int_{-\infty}^{\infty} dx e^{-2\varphi(x)} \int_{-\infty}^{x} e^{2\varphi} + e^{-\varphi(\infty)} \int_{-\infty}^{\infty} dx e^{2\varphi(x)} \int_{-\infty}^{x} e^{-2\varphi} = C_1,$$

$$e^{\varphi(\infty)} \int_{-\infty}^{\infty} dx e^{-2\varphi(x)} \int_{\infty}^{x} dy e^{2\varphi(y)} \int_{-\infty}^{y} dz e^{-2\varphi(z)} \int_{-\infty}^{z} e^{2\varphi} +$$

$$+ e^{-\varphi(\infty)} \int_{-\infty}^{\infty} dx e^{2\varphi(x)} \int_{-\infty}^{x} dy e^{-2\varphi(y)} \int_{-\infty}^{y} dz e^{2\varphi(z)} \int_{-\infty}^{z} e^{-2\varphi} = C_2, \ldots$$

Remark. It could be interesting to compute the Poisson brackets between the monodromy operators of the $\partial^2 + u + \lambda$, with Poisson brackets of u corresponding to the higher GD structures. In the case of GD1 the Poisson brackets between monodromies are given by the rational r-matrix $\frac{t}{\lambda-\mu}$ on the group $SL_2(C[[\lambda]])$ (Faddeev-Takhtajan), in the case of GD2 they are given by the trigonometric r-matrix $\frac{1}{2}\frac{\lambda+\mu}{\lambda-\mu}t + r$ (in the Belavin-Drinfeld notations). These two Poisson structures on $SL_2(C[[\lambda]])$ are compatible.

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1. M.Adler, Invent. Math. **50**, 219 (1979).
2. I.M.Gel'fand, L.A.Dikii, preprint IPM (Moscow) 136, in Russian, (1979).
3. A.A.Kirillov, Lect. Notes in Math. **970**, 101 Springer (1982).
4. F.Magri, Lect. Notes in Physics **120**, 233 Springer (1980).
5. A.Weinstein, Soc. Math. Fr., Astérisque, hors série (1985) 421.