MATRIX VERSIONS OF THE CALOGERO MODEL

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Matrix generalizations of the N-particle quantum-mechanical Calogero model classifying according to representations of the symmetric group S_N are considered. Symmetry properties of the eigen-wave functions in the matrix Calogero models are analyzed.

The Calogero model is the N-particle quantum-mechanical model on a line with the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N}\left[-d_i^2 + x_i^2\right] + g\sum_{j<i}^{N}(x_i - x_j)^{-2}$$

where $d_i = \frac{\partial}{\partial x_i}$. It gives a prime example of a solvable N-body quantum mechanical model [1,2], which has interesting physical applications. In particular, it is closely related to the matrix models [3,4], while the generalized differential operators which underly integrability of the model appear in a number of quite different problems such as, e.g., the decoupling equations in certain formulations of conformal models [5] - [8] and the problem of quantization on the sphere and hyperboloid [9]. It was shown in [10]-[13] that the Calogero model can be interpreted as the one-dimensional reduction of the full anyon problem. Recently it was argued [14] that the Calogero model (and its trigonometric generalization) can be identified with the $2D$ Yang-Mills theory on a cylinder. Another intriguing link is that the higher-spin gauge theories in three [15] and four [16] space-time dimensions exhibit infinite-dimensional symmetries, higher-spin symmetries, described by the algebra of observables of the Calogero model.

One can speculate that the reason for all these links of the Calogero model is that the algebraic structures underlying it are as fundamental as those of the ordinary harmonic oscillator. The matrix generalizations of the Calogero model we focus on in this letter may get useful applications as well.

The spectrum of the Hamiltonian H was found by Calogero [1]. The singularities of H at the planes $x_i = x_j$ force one to consider the wave functions of H separately in $N!$ distinct domains singled out by the sets of inequalities $x_{i_1} < x_{i_2} < \ldots < x_{i_N}$. Performing the similarity transformation $\Psi = \beta^\nu \Phi$, where $\beta = \prod_{x_i > x_j}(x_i - x_j)$, so that for $g = \nu(\nu - 1)$ the transformed Hamiltonian $H_{Cal} = \beta^\nu H \beta^{-\nu}$ takes the form

$$H_{Cal} = -\frac{1}{2}\sum_{i=1}^{N}\left[d_i^2 - x_i^2 + \nu\sum_{j\neq i}\frac{2}{x_i - x_j}d_i\right], \qquad (1)$$

Calogero argued [1] that, for $\nu > 0$, regular eigenfunctions $H_{Cal}\Phi_n = E_n\Phi_n$ are of the form

$$\Phi_n = \phi_{nk}(r)P_k(x)\,,\ r^2 = \frac{1}{N}\sum_{i<j}(x_i - x_j)^2$$

where $P_k(x)$ are homogeneous polynomials of degree k obeying the "generalized harmonic equation"

$$\left(\sum_{i=1}^{N} d_i^2 + \nu\sum_{i\neq j}\frac{1}{x_i - x_j}(d_i - d_j)\right) P_k = 0 \qquad (2)$$

while $\phi_{nk}(r)$ obeys certain equation [1] which fixes energy spectrum E_n. Calogero proved that every polynomial obeying (2) is some symmetrical polynomial of x_i. One interpretation of this elegant result is that the model automatically selects the subspace of totally symmetric wave functions which extend to the whole coordinate space.

More recently, it was shown [17] how one can construct the set of eigen-wave functions for the Calogero model, (thus describing solutions of (2)), with the aid of the approach based on the permutation operators K_{ij} interchanging the coordinates x_i and x_j. The basic point is that by introducing

$$D_i = d_i + \nu\sum_{j\neq i}\frac{1}{x_i - x_j}(1 - K_{ij}) \qquad (3)$$

with K_{ij} obeying the properties $K_{ij}x_j = x_iK_{ij}$, $K_{ij}K_{jl} = K_{il}K_{ij} = K_{jl}K_{il}$, $K_{ij} = K_{ji}$, $(K_{ij})^2 = 1$, one observes, first, [19, 18, 17] that $[D_i, D_j] = 0$ and, second, that one can define [17] such creation and annihilation operators $a_i^{\mp} = \frac{1}{\sqrt{2}}(x_i \pm D_i)$, obeying the commutation relations

$$[a_i^{\pm}, a_j^{\pm}] = 0\,, \qquad [a_i^-, a_j^+] = \delta_{ij}(1 + \nu\sum_l K_{il}) - \nu K_{ij} = A_{ij}\,, \qquad (4)$$

that the Hamiltonian $H_{Univ} = \frac{1}{2}\sum_i \{a_i^+, a_i^-\}$ fulfills the basic property

$$[H_{Univ}, a_i^{\pm}] = \pm a_i^{\pm} \qquad (5)$$

and turns out to be related with the original Calogero Hamiltonian (1) in the following simple way

$$H_{Univ} = H_{Cal} + \frac{1}{2}\nu\sum_{j\neq i}\frac{1}{(x_i - x_j)^2}(1 - K_{ij})\,. \qquad (6)$$

Based on (5) one easily constructs [17] the set of eigen-wave functions of the universal Calogero Hamiltonian H_{Univ} (6) via the standard procedure by defining the ground state through $a_i^-|0\rangle = 0$, $K_{ij}|0\rangle = |0\rangle$. [1)] Since the second term on the r.h.s. of (6) trivializes for totally symmetric states, one observes

[1)] It should be noted that restriction $\nu > -\frac{1}{N}$ leads to the vacuum $|0>$ with a finite norm and therefore the restriction $\nu > 0$ guarantees the finiteness of the norm of vacuum for every N. A detailed consideration of more delicate cases with non-normalizable vacua and small ν when some singular wave functions may also exist will be given elsewhere.

that for this case H_{Univ} amounts to H_{Cal}, thus recovering the totally symmetric wave functions of the Calogero model. It is worth mentioning that the universal Calogero Hamiltonian H_{Univ} is well defined for the wave functions having arbitrary symmetry properties while the original Hamiltonian (1) makes sense only when it coincides with H_{Univ} , i.e. when the second term on the r.h.s. of (6) trivializes.

The question we address in this paper is whether there exist other quantum-mechanical models which amount to the universal Calogero model (6) for subspaces of wave functions corresponding to one or another Young diagram. We show that such quantum-mechanical models are described by the Hamiltonians

$$\hat{H}_{Cal} = H_{Cal} I + \frac{1}{2}\nu \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}(I - T_{ij}) . \tag{7}$$

Here I is the unit $m \times m$ matrix and $T_{ij} = T(p_{ij})$ where T is some $m \times m$ matrix unitary representation of the symmetric group with elementary $i \leftrightarrow j$ permutations p_{ij}. Since $T_{ij}^2 = I$, each unitary matrix T_{ij} is hermitian, that ensures hermiticity of the Hamiltonian (7). [2] It is worth mentioning that both supersymmetric [20, 13] and matrix [21] generalizations of the Calogero Hamiltonian considered previously correspond to particular cases of (7) for some (reducible) representations T of the symmetric group.

The Calogero result on the symmetry property of the wave functions extends to the Hamiltonian (7) as follows: non-singular eigenfunctions of the Hamiltonian (7) with $\nu > 0$ exist if they obey the conditions

$$K_{ij}\Phi = T_{ij}\Phi \tag{8}$$

for all i and j, i.e. when the Hamiltonians (6) and (7) coincide.

This fact is the main result of this paper which implies in particular that the condition (8) which was shown in [21, 13] to be convenient to impose to solve the problem is a sort of necessary condition which cannot be avoided. Effectively this means that the action of the symmetric group S_N on the coordinates x_i generated by K_{ij} realizes the same representation of S_N as T_{ij} do. (In particular when $T_{ij} = 1$ the wave functions turn out to be symmetrical.) The proof of this statement can be given as follows:

(i) one easily checks that if Φ (which is m-column) is some solution of

$$\hat{H}_{Cal}\Phi = E\Phi \ , \tag{9}$$

then $K_{ij}T_{ij}\Phi$ and therefore $Q_{ij} = (1 - K_{ij}T_{ij})\Phi$ are some its solution either;

(ii) multiplying the both sides of (9) by $(x_i - x_j)^2$ and setting $x_i = x_j$ one observes that $(1 - T_{ij})\Phi \,|_{x_i = x_j} = 0$. Since K_{ij} trivializes for $x_i = x_j$, one concludes that $Q_{ij}\,|_{x_i=x_j} = 0$.

(iii) using this one then proves along the lines of the original Calogero proof that $Q_{ij} = 0$. The main steps are as follows. Suppose that $Q_{ij} \neq 0$. Then $Q_{ij} = (x_i - x_j)^l R_{ij}$ with some positive integer l and the column $R_{ij}\,|_{x_i=x_j} \neq 0$. Substituting this back into (9), taking in account (i), and analyzing the lowest order terms in $(x_i - x_j)$ one gets that

$$\left(l(l-1) + 2\nu l - \nu(1 - T_{ij})\right) R_{ij}\,|_{x_i=x_j} = 0 . \tag{10}$$

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[2] To avoid misunderstandings, let us emphasize that, in contrast to K_{ij}, $m \times m$ x-independent matrices T_{ij} commute with x_i and d_i.

Now one observes that $T_{ij}K_{ij}Q_{ij} = -Q_{ij}$. Therefore $T_{ij}K_{ij}R_{ij} = -(-1)^l R_{ij}$ and $T_{ij}R_{ij}\ |_{x_i=x_j} = -(-1)^l R_{ij}\ |_{x_i=x_j}$. Hence either $\left(l(l-1)+2\nu l - \nu(1+(-1)^l)\right) = 0$ which is impossible for $l > 0$ and $\nu > 0$, or $R_{ij}\ |_{x_i=x_j}= 0$, thus completing the proof.

Thus, entire eigen-wave functions of the Hamiltonian (7) can exist only when $Q_{ij} = 0$ and therefore (8) is true. From the results of [17] it follows that such solutions of (9) do exist. Actually, let T be some irreducible representation of the symmetric group S_N described by an appropriate Young tableaux. Let $R(a_i^+)$ be m-column of homogeneous polynomials of a_i^+ of degree k satisfying the condition $K_{ij}RK_{ij}^{-1} = T_{ij}R$ for all i and j. It is evident that for every m-dimensional representation T of S_N there exist such sufficiently large k that such a column $R(a_i^+)$ exists. Applied to the symmetric groundstate of the Hamiltonian (6) it gives some solution of (9): $\hat{H}_{Cal}R|0\rangle = H_{Univ}R|0\rangle = (RH_{Univ} + kR)|0\rangle = (E_0 + k)R|0\rangle$.

The general structure of creation and annihilation operators for the Hamiltonian (7) is as follows. Any annihilation operator A_n is $m \times m$-matrix operator mapping eigenfunctions of (9) with some eigenvalue E to eigenfunctions having the eigenvalue $E - n$. Matrices A_n have to preserve (8), i.e. if a vector-function Φ belongs to the space of linear combinations of eigenfunctions of (9) then $K_{ij}A_n\Phi = T_{ij}A_n\Phi$ and hence the restriction of A_n (which we will identify with A_n) to this space satisfy the following conditions

$$K_{ij}A_nK_{ij}^{-1} = T_{ij}A_nT_{ij}^{-1} \tag{11}$$

for any i and j. The defining relation $[\hat{H}_{Cal}, A_n] = -nA_n$ along with (11) leads to

$$[H_{Univ},\ A_n] = -nA_n \tag{12}$$

Since H_{Univ} is proportional to the unit matrix I the relation (12) is true for all elements of matrix A_n separately and one concludes that n-degree annihilation operators A_n for the Hamiltonian (9) are matrices obeying (11) with elements depending on a_i and a_i^+ each having grading $-n$. We hope to present a constructive description of A_n elsewhere.

One can speculate that the models under investigation describe interactions of several groups of particles with abnormal mutual statistics. For the general case, various types of interacting matrix Calogero models are classified according to irreducible representations of the symmetric group S_N.

As a simplest example let us consider the case corresponding to the Young diagram with two rows containing $N - 1$ boxes and 1 box respectively. It is convenient to describe the space of this representation via column-vectors with the components

$$\Phi_i = (1 - K_{iN})F\ ,\ \ i = 1, ..., N-1 \tag{13}$$

where F is some function symmetric under transpositions of the first $N - 1$ variables: $K_{ij}F = F$ for $i, j = 1, 2, ..., N - 1$.

The action of the operators K_{ij} on this columns is the same as the action of some $(N-1)\times(N-1)$ matrices $\tilde{T}_{ij}$, $(K_{ij}\Phi)_l = \sum_{k=1}^{N-1}(\tilde{T}_{ij})_{lk}\Phi_k$:

$$\begin{aligned} K_{ij}\Phi_l = \Phi_l\ ,\ \ &when\ \ i,j,l = 1,2,...,N-1\ ,\ \ i \neq l\ ,\ j \neq l \\ K_{ij}\Phi_j = \Phi_i\ ,\ \ &when\ \ i,j = 1,2,...,N-1 \\ K_{iN}\Phi_i = -\Phi_i\ ,\ \ &i = 1,2,...,N-1 \\ K_{iN}\Phi_j = \Phi_j - \Phi_i\ ,\ \ &when\ \ i,j, = 1,2,...,N-1\ ,\ \ i \neq j \end{aligned} \tag{14}$$

The matrices $\tilde{T}_{ij}$ are equivalent to unitary matrices $T_{ij} = Q^{-1}\tilde{T}_{ij}Q$ where Q is any matrix satisfying the conditions

$$\sum_i (Q_{ki})^2 = 2 \quad , \quad \sum_i (Q_{ki} - Q_{li})^2 = 2 \qquad k \neq l \tag{15}$$

i.e. its $N-1$ rows together with zero can be interpreted as coordinates of the apices of some rectilinear N-hedron in $(N-1)$-dimensional space. For example, one can fix $Q_{ij} = \delta_{ij} - \frac{1}{N-1}(1+\sqrt{N})$, $(Q^{-1})_{ij} = \delta_{ij} - \frac{1}{N-1}(1+\frac{1}{\sqrt{N}})$.

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