Письма в ЖЭТФ, том 58, вып.11, стр.925 - 929 ©1993 г. 10 декабря

CAUSALITY AND KÄLLEN-LEHMANN-LIKE REPRESENTATION OF THE FERMIONIC STRING PROPAGATOR[1)]

V.Ya.Fainberg[†*], *A.L.Fil'kov*[*]

[†]*Physical Department, Middle East Technical University 06531 Ankara, Turkey*

[*]*Department of Theoretical Physics, P.N.Lebedev Physical Institute 117924, Moscow, Russia*

Submitted 2 November 1993

A spectral representation for the Green functions of the free fermionic Ramond-Neveu-Schwarz string is obtained. It is valid for any initial and final string configurations and manifests an exponential growth of spectral densities intrinsic in nonlocalizable theories. Causal and singular properties of the propagators are considered.

The Källen-Lehmann-like representation for string propagators with point-like boundary conditions was obtained in [1,2], starting from the path integral approach [3]. The case of any initial and final string configurations for bosonic strings was considered in [4] by means of the operator formalism. As it should be expected [5], in all these cases the growth of the corresponding spectral densities of the propagators turns out to be linear exponential with the fundamental length $l = 2\pi\sqrt{\alpha'}$. Field theories with such a growth of matrix elements are nonlocalizable and were investigated in [6,7].

The purpose of the present work is to derive such a spectral representation for the fermionic Ramond-Neveu-Schwarz string propagators and to show that causal and singular properties of the propagators do not depend on boundary conditions.

At the beginning we will consider the NS-sector. We will follow the notations of [8] and use a system of units for which $2\alpha' = 1$.

Let us proceed from the following BRST-invariant expression for the Hamiltonian of the open fermionic string in the NS-sector:

$$\hat{L}_0 = \frac{1}{2}\hat{p}^2 + \sum_{n=1}^{\infty} n(\hat{a}_{-n}\hat{a}_n + \hat{c}_{-n}\hat{b}_n + \hat{b}_{-n}\hat{c}_n) + \sum_{r=\frac{1}{2}}^{\infty} r(\hat{\psi}_{-r}\hat{\psi}_r + \hat{\beta}_{-r}\hat{\gamma}_r - \hat{\gamma}_{-r}\hat{\psi}_r) - \frac{1}{2} \ , \quad (1)$$

where the operators obey the following commutation relations:

$$[\hat{x}^\mu, \hat{p}^\nu]_- = i\delta^{\mu\nu} \qquad [\hat{a}^\mu_m, \hat{a}^\nu_n]_- = \mathrm{sign}(m)\delta_{m+n}\delta^{\mu\nu} \qquad [\hat{\psi}^\mu_r, \hat{\psi}^\nu_s]_+ = \delta_{r+s}\delta^{\mu\nu},$$

$$[\hat{c}_m, \hat{b}_n]_+ = \delta_{m+n}, \qquad [\hat{\gamma}_r, \hat{\beta}_s]_- = \delta_{r+s}, \qquad \mu,\nu = 1,\dots,D = 10 \ , \quad (2)$$

with the other (anti)commutators being equal to zero.

We use a picture in which all the operators with the negative (positive) indices are the creation (annihilation) operators. It is convenient to use the holomorphic representation of these commutation relations with the following integration measure (the index μ is omitted):

$$1 = \int d\omega = \prod_{n=1}^{\infty} \int \frac{da_n^* da_n}{2\pi i} e^{-a_n^* a_n} \int dc_n^* db_n db_n^* dc_n e^{-c_n^* b_n - b_n^* c_n} \times$$

[1)]Work supported in part by the Russian Foundation of Fundamental Investigations grant 93-02-3379

$$\times \prod_{r=\frac{1}{2}}^{\infty} \int d\psi_r^* d\psi_r e^{-\psi_r^*\psi_r} \int \frac{d\gamma_r^* d\gamma_r d\beta_r^* d\beta_r}{(2\pi i)^2} e^{-\beta_r^*\gamma_r + \gamma_r^*\beta_r} \ . \tag{3}$$

The Green function for the operator $\hat{L}_0$ obeys the equation:

$$\hat{L}_0 G(x, A^*; x', A') = \delta(x, A^*; x', A') \ . \tag{4}$$

Here x and x' are positions of the string center of mass, A^* and A' are sets of all the oscillator variables and $\delta(x, A^*; x', A')$ stands for the kernel of the unity operator:

$$\delta(x, A^*; x', A') = \delta^{10}(x - x') \sum_k \Delta_k(A^*, A') \equiv$$

$$\equiv \delta^{10}(x - x') \sum_k (-)^{k^\gamma} \prod_{n=1}^{\infty} \Big\{ (c_n^* b_n')^{k_n^c} (b_n^* c_n')^{k_n^b} \prod_{\mu=1}^{10} \frac{(a_{n\mu}^* a_{n\mu}')^{k_{n\mu}^a}}{k_{n\mu}^a!} \Big\} \times$$

$$\times \prod_{r=\frac{1}{2}}^{\infty} \Big\{ \frac{(\gamma_r^* \beta_r')^{k_r^\gamma}}{k_r^\gamma!} \frac{(\beta_r^* \gamma_r')^{k_r^\beta}}{k_r^\beta!} \prod_{\mu=1}^{10} (\psi_{r\mu}^* \psi_{r\mu}')^{k_{r\mu}^\psi} \Big\} \ , \tag{5}$$

where k is some multi-index connected with the numbers of occupation. A solution of the equations (4), (5) can be represented in the form:

$$G(x, A^*; x', A') = 2 \sum_k \int \frac{d^{10}p}{(2\pi)^{10}} \frac{e^{ip(x-x')}}{p^2 + 2[k] - 1} \Delta_k(A^*, A'),$$

$$[k] = \sum_{n=1}^{\infty} n k_n + \sum_{r=\frac{1}{2}}^{\infty} r k_r \ . \tag{6}$$

Actually this expression is somewhat formal because there is a singularity connected with the presence of the tachyon in the mass operator spectrum. This is a defect of the theory and we will not discuss it but note that this difficulty is absent in the R-sector.

Using a summation over the mass operator spectrum, we can reduce this representation to the Källen-Lehmann form:

$$G(x, A^*; x', A') = \sum_{M^2} \int \frac{d^{10}p}{(2\pi)^{10}} \frac{e^{ip(x-x')}}{p^2 + M^2} \rho(M^2; A^*, A') \ , \tag{7}$$

where the spectral density matrix is:

$$\rho(M^2; A^*, A') = \int_{-1}^{1} d\phi e^{i\pi\phi(M^2+1)} \sum_k e^{-2i\pi[k]\phi} \Delta_k(A^*, A') \ . \tag{8}$$

Taking a trace over all the oscillator variables, we obtain the spectral density for the fixed x and x' only:

$$\rho(M^2) = \mathrm{Sp}\rho(M^2; A^*, A') = \int_{-1}^{1} d\phi e^{i\pi\phi(M^2+1)} \prod_{n=1}^{\infty} \left[\frac{1 + e^{-i\pi\phi(2n-1)}}{1 - e^{-2i\pi\phi n}} \right]^8 \ . \tag{9}$$

The asymptotic behavior of $\rho(M^2)$ is known to be linear exponential [1,2,8]. For more general boundary conditions we can use the method of [4]. If $\rho \equiv \rho(M^2; A^*, A')$ is a compact operator and $R(A^*, A')$ is a bounded one in the Hilbert space, therefore $|\mathrm{Sp}\rho| \leq ||\rho||_1$ and $|\mathrm{Sp}R\rho| \leq ||\rho||_1 ||R||$, where $||\rho||_1$ is the nuclear norm of the operator ρ. It can be shown that in our case the nuclear norm of ρ coincides with $\rho(M^2)$. Therefore the spectral density $\mathrm{Sp}R\rho$ behaves as a linear exponent for any bounded R in the Källen-Lehmann-like representation (7).

If we choose the initial and final states in the following form:

$$\Phi(X^{i,f}; A^*) = \prod_{n=1}^{\infty} \left\{ \frac{1}{\sqrt{2\pi^5}} \exp(c_n^* b_n^*) \prod_{\mu=1}^{10} \exp\left(-\frac{1}{2}(X_{n\mu}^{i,f})^2 - i\sqrt{2} X_{n\mu}^{i,f} a_{n\mu}^* + \frac{1}{2}(a_{n\mu}^*)^2\right) \right\} \quad (10)$$

and take a trace over the other oscillator variables we obtain for the spectral density:

$$\rho(M^2; X^f, X^i) = N^{-1} \int_{-1}^{1} d\phi e^{i\pi\phi(M^2+1)} \prod_{n=1}^{\infty} \frac{\left(1 + e^{-i\pi\phi(2n-1)}\right)^8}{\left(1 - e^{-4i\pi\phi n}\right)^4} \times$$

$$\times \exp\left\{ i \sum_{n=1}^{\infty} \frac{\left((X_n^f)^2 + (X_n^i)^2\right) \cos(2\pi n\phi) - 2X_n^f X_n^i}{2\sin(2\pi n\phi)} \right\}, \qquad N = \prod_{n=1}^{\infty} 2\pi^5. \quad (11)$$

This result corresponds to the case when initial and final string positions in the space-time are fixed. In this case the Green function can be written down in the form $(\Delta x \equiv x - x')$:

$$G(X^f, X^i) = \frac{1}{32N\pi^9} \int_0^{\infty} \frac{dT}{T^5} e^{\pi T/2} \prod_{n=1}^{\infty} \frac{\left(1 + e^{-\pi T(n-1/2)}\right)^8}{\left(1 - e^{-2\pi n T}\right)^4} \times$$

$$\times \exp\left\{ -\frac{(\Delta x)^2}{2\pi T} - \sum_{n=1}^{\infty} \frac{\left((X_n^f)^2 + (X_n^i)^2\right) \cosh(\pi n T) - 2X_n^f X_n^i}{2\sinh(\pi n T)} \right\}, \quad (12)$$

which coincides with the result of the path integration [1,2].

Let us consider the R-sector. Now the equation for the Green function is:

$$\hat{F}_0 G(x, A^*; x', A') = \mathrm{I}\delta(x, A^*; x', A'), \quad (13)$$

where

$$\hat{F}_0 = \frac{1}{i\sqrt{2}} \hat{p}\Gamma + \Gamma_{11} \Big[\sum_{n=1}^{\infty} \{ \sqrt{n}(\hat{a}_{-n}\hat{\psi}_n + \hat{\psi}_{-n}\hat{a}_n) - 2(\hat{b}_{-n}\hat{\gamma}_n + \hat{\gamma}_{-n}\hat{b}_n) -$$

$$- \frac{1}{2} n(\hat{\beta}_{-n}\hat{c}_n - \hat{c}_{-n}\hat{\beta}_n) \} - 2\gamma_0 b_0 \Big] \quad (14)$$

and I is an unity matrix. A solution of (13) can be represented in the form:

$$G(x, A^*; x', A') = \sum_{M^2} \int \frac{d^{10}p}{(2\pi)^{10}} \frac{e^{ip(x-x')}}{p^2 + M^2} \left[p\Gamma \rho_1(M^2; A^*, A') + \Gamma_{11}\rho_2(M^2; A^*, A') \right], \quad (15)$$

where the spectral densities are:

$$\rho_1(M^2; A^*, A') = -\frac{i}{\sqrt{2}} \int_{-1}^{1} d\phi e^{i\pi\phi M^2} \sum_k e^{-2i\pi[k]\phi} \Delta_k(A^*, A'),$$

$$\rho_2(M^2;A^*,A') = \int_{-1}^{1} d\phi e^{i\pi\phi M^2} \sum_{m=1}^{\infty} e^{-2i\pi m\phi} \{\sqrt{m}(a_m^* \psi_m' + \psi_m^* a_m') -$$

$$-2(b_m^* \gamma_m' + \gamma_m^* b_m') - \frac{1}{2} m(\beta_m^* c_m' - c_m^* \beta_m')\} \sum_k e^{-2i\pi[k]\phi} \Delta_k(A^*,A') \ . \qquad (16)$$

Taking a trace over all the oscillator variables, we obtain:

$$\rho_1(M^2) = \mathrm{Sp}\rho_1(M^2;A^*,A') = -\frac{i}{\sqrt{2}} \int_{-1}^{1} d\phi e^{i\pi\phi M^2} \prod_{n=1}^{\infty} \left[\frac{1+e^{-2i\pi\phi n}}{1-e^{-2i\pi\phi n}} \right]^8,$$

$$\rho_2(M^2) = \mathrm{Sp}\rho_2(M^2;A^*,A') = 0 \ . \qquad (17)$$

Choosing the boundary conditions (10), we obtain:

$$\rho_2(M^2;X^f,X^i) = 0,$$

$$\rho_1(M^2;X^f,X^i) = -\frac{i}{\sqrt{2}} N^{-1} \int_{-1}^{1} d\phi e^{i\pi\phi M^2} \prod_{n=1}^{\infty} \left(\frac{1+e^{-2i\pi\phi n}}{1-e^{-2i\pi\phi n}} \right)^4 \times$$

$$\times \exp\left\{ i \sum_{n=1}^{\infty} \frac{((X_n^f)^2 + (X_n^i)^2)\cos(2\pi n\phi) - 2X_n^f X_n^i}{2\sin(2\pi n\phi)} \right\}, \qquad N = \prod_{n=1}^{\infty} 2\pi^5; \qquad (18)$$

$$G(X^f,X^i) = -\frac{1}{32\sqrt{2}N\pi^9} \left(\Gamma \frac{\partial}{\partial x} \right) \int_0^{\infty} \frac{dT}{T^5} \prod_{n=1}^{\infty} \left[\frac{1+e^{-\pi nT}}{1-e^{-\pi nT}} \right]^4 \times$$

$$\times \exp\left\{ -\frac{(\Delta x)^2}{2\pi T} - \sum_{n=1}^{\infty} \frac{((X_n^f)^2 + (X_n^i)^2)\cosh(\pi nT) - 2X_n^f X_n^i}{2\sinh(\pi nT)} \right\} \ . \qquad (19)$$

The formula (19) coincides with the path integration result [1,2].

Using the results obtained for the NS-sector, one can show that the growth of the spectral densities is linear exponential for any boundary conditions.

Let us discuss the singular properties of the propagators (12) and (19). Apart from the infrared tachyon singularity in the NS-sector both these expressions possess the ultraviolet ($T \to 0$) singularities connected with the exponential growth of the spectral densities. Taking into account the properties of the θ-functions (see, for example, [8]), we can obtain the following equations:

$$e^{\pi T/2} \prod_{n=1}^{\infty} \frac{\left(1+e^{-\pi T(n-1/2)}\right)^8}{\left(1-e^{-2\pi nT}\right)^4} = T^2 e^{\pi/T} \prod_{n=1}^{\infty} \frac{\left(1+e^{-\frac{2\pi(2n-1)}{T}}\right)^8}{\left(1-e^{-\frac{2\pi n}{T}}\right)^4} \qquad (20)$$

for the NS-sector and

$$\prod_{n=1}^{\infty} \frac{\left(1+e^{-\pi nT}\right)^8}{\left(1-e^{-2\pi nT}\right)^4} = \frac{T^2}{16} e^{\pi/T} \prod_{n=1}^{\infty} \frac{\left(1-e^{-\frac{2\pi(2n-1)}{T}}\right)^8}{\left(1-e^{-\frac{2\pi n}{T}}\right)^4} \qquad (21)$$

for the R-sector. It can be easily seen that the main ultraviolet singularity is proportional to

$$\int_0 \frac{dT}{T^3} \exp\left[\frac{\pi}{T} - \frac{(\Delta x)^2}{4\pi\alpha' T}\right] \quad \text{for the NS-sector} \tag{22}$$

and

$$\frac{\partial}{\partial x} \int_0 \frac{dT}{T^3} \exp\left[\frac{\pi}{T} - \frac{(\Delta x)^2}{4\pi\alpha' T}\right] \quad \text{for the R-sector.} \tag{23}$$

This confirms the general conclusion obtained in [1] (see the formula (4.56)).

To study the string propagator causal properties it is necessary to pass to the Minkowski space-time by means of the analytic continuation in p_0. Strictly speaking, it is possible if we forget about the tachyon state in the NS-sector. Then we can see from (7), (15), (22) and (23) that the propagators (12), (19) have noncausal singularities in the space-like region $(\Delta x)^2 = (\Delta \mathrm{x})^2 - (\Delta x_0)^2 \geq 4\pi^2\alpha'$ and this region does not increase if more general boundary conditions are chosen.

For the closed fermionic string the Källen-Lehmann-like representation can also be derived. Formulae like (7), (12) for the NS-NS-sector, like (15), (16), (19) for the R-NS-sector and more complicated for the R-R-sector will be written down in another paper.

In conclusion we would like to note that the question of the minimal region of nonlocality in string theory is closely connected with the remarkable role of the modular invariance in the vanishing of the ultraviolet divergences. We do not still know how to sew the ends of the closed string propagator in order to get the fundamental region of the integration over modular parameters automatically [8]. This is a task of a future field theory of closed strings.

We are grateful to M.A.Soloviev for fruitful discussions.

---

1. V.Ya.Fainberg and A.V.Marshakov, Phys.Lett. B **211**, 82 (1988); Proc. Lebedev Phys. Inst. **201**, 139–179 (1990).
2. A.V.Marshakov, Nucl. Phys. B **312**, 178 (1989).
3. A.M.Polyakov, Phys. Lett. B **103**, 207,211 (1981).
4. V.Ya.Fainberg and M.A.Soloviev, Theor. Math. Phys. **93**, 1438 (1992).
5. S.Fubini and G.Veneziano, Nuovo Cim. A **64**, 811 (1969); K.Bordakci and S.Mandelstam, Phys. Rev. **184**, 1640 (1969).
6. M.Z.Iofa and V.Ya.Fainberg, Sov. Phys. JETP **29**, 880 (1969); Theor. Math. Phys. **1**, 143 (1969).
7. V.Ya.Fainberg and M.A.Soloviev, Ann. of Phys. **113**, 421 (1978).
8. M.B.Green, J.H.Schwarz, and E.Witten, "Superstring theory", (1987), Cambridge Univ. Press, Cambridge.