NEW RESTRICTIONS ON SPATIAL TOPOLOGY OF THE UNIVERSE FROM MICROWAVE BACKGROUND TEMPERATURE FLUCTUATIONS

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If the Universe has the topology of a 3-torus (T^3), then it follows from recent data on large-scale temperature fluctuations $\Delta T/T$ of the cosmic microwave background that either the minimal size of the torus is at least of the order of the present cosmological horizon, or the large-scale $\Delta T/T$ pattern should have a symmetry plane (in case of the effective T^1 topology) or a symmetry axis (in case of the effective T^2 topology), the latter possibility being probably excluded by the data.

The possibility that a spacelike hypersurface $t = const$ (or $\varepsilon = const$) of the Friedmann-Robertson-Walker (FRW) cosmological model may possess a non-trivial (i.e. not R^3) topology was known long ago in connection with the S^3 topology in the case of a closed Universe ($\Omega_{tot} \equiv \varepsilon_{tot}/\varepsilon_{crit} > 1$). In this case, it follows from the age of the Universe and direct dynamical estimates of matter density that the corresponding topological size - the radius of the Universe - cannot be significantly less than the present cosmological horizon. A way to get small topological lengths is to assume a spatially-flat 3-space with the 3-torus topology (T^3), i.e. to make the following identifications of points of the 3-D flat hypersurface $t = const$:

$$x \equiv x + L_1,\ y \equiv y + L_2,\ z \equiv z + L_3, \quad L_1 \geq L_2 \geq L_3 . \tag{1}$$

This possibility was considered in [1-3] with the conclusion that aL_1 should exceed $500 - 1000$ Mpc where $a \equiv a(t_0)$ is the present scale factor of the Universe (in this paper, all scales are given for $H = 50$ km/s/Mpc). If $aL_1 \gg R_{hor}$ where R_{hor} is the present size of the cosmologocal horizon ($R \approx 12000$ Mpc if $a(t) \propto t^{2/3}$ now), then the effective spatial topology of the Universe becomes T^2; for $aL_2 \gg R_{hor}$, it is effectively T^1. Note also that the remaining possibility to have a non-trivial topology with small characteristic sizes in an open FRW Universe [4] is practically excluded because it requires $\Omega_{tot} \ll 1$ that contradicts observational data ($\Omega_m = 0.2 - 0.4$ from dynamical estimates based on optical galaxies; use of IRAS galaxies yields a larger value for Ω_m).

Classical general relativity does not put any restrictions on L_i, they are just a part of initial conditions for the Universe at the singular hypersurface $a = t = 0$. Different versions of the cosmological scenario with a de Sitter (inflationary) phase in the Early Universe [5] do not preclude this topology, too. In this case, the T^3 topology may be a result of quantum creation of the Universe [6] (if this process is believed to be properly described by a WKB solution of the Wheeler-De Witt equation exponentially damping under a potential barrier in the direction from $a = 0$ to the border of a classically allowed region). Then, however, as a result of inflation, the present topological sizes aL_i are expected to be much more than R_{hor} with an overwhelming probability. It is also possible to generate a non-trivial topology of the $\varepsilon = const$ hypersurface during inflation [7] but not of the T^3 type. Thus, the prediction of the inflationary scenario is that spatial topology of the Universe should be trivial at all observable scales up to R_{hor} and even larger.

Irrespective of any assumptions of the Early Universe behaviour, it becomes possible now to strongly restrict the hypothesis of the spatial T^3 topology of the Universe directly from recent observational data [8,9] on large-scale fluctuations of the cosmic microwave background radiation [1)]. Here, the COBE data [9] appear to be the most appropriate. If $\Delta T/T$ fluctuations are produced by adiabatic perturbations and $a(t) \propto t^{2/3}$ since decoupling till the present time (the matter-dominated regime in a spatially flat FRW Universe), then the Sachs-Wolfe formula [11] is valid for large angles $\vartheta \geq 2^\circ$ ($l \leq 30$):

$$\frac{\Delta T}{T}(\theta, \varphi) = \frac{1}{3}\Phi(R_{rec}/a, \theta, \varphi) = -\frac{1}{10}h(R_{rec}/a, \theta, \varphi) \tag{2}$$

where $\Phi(\vec{r})$ - the gravitational potential - defines a perturbed space-time metric in the longitudinal gauge:

$$ds^2 = (1 + 2\Phi)dt^2 - a^2(t)(1 - 2\Phi)(dx^2 + dy^2 + dz^2)\,, \tag{3}$$

$h(\vec{r})$ is a metric perturbation in the ultra-synchronous gauge (where $h^\beta_\alpha \approx h(\vec{r})\delta^\beta_\alpha$ as $t \to 0$, $\alpha, \beta = 1, 2, 3$) and R_{rec} is the present radius of the surface of last scattering, $R_{rec} \approx 0.97 R_{hor}$. Φ and h do not depend on t in linear approximation.

In a 3-torus Universe, the spectrum of Fourier modes is discrete:

$$\Phi(\vec{r}) = \sum_{npq} \Phi(\vec{k}) e^{i\vec{k}\vec{r}}, \quad \vec{k} = \left(\frac{2\pi n}{L_1}, \frac{2\pi p}{L_2}, \frac{2\pi q}{L_3}\right), \quad n, p, q \in \mathbf{Z}, \quad n^2 + p^2 + q^2 \neq 0 \tag{4}$$

(the $n = p = q = 0$ mode is a gauge one). $\Phi(\vec{k})$ is not assumed to have the flat (Harrison-Zeldovich) spectrum (contrary to [10]). Expanding $e^{i\vec{k}\vec{r}}$ in terms of normalized spherical harmonics, we get:

$$\frac{\Delta T}{T} = \sum_{lm} \left(\frac{\Delta T}{T}\right)_{lm} Y_{lm}(\theta, \varphi), \quad \left(\frac{\Delta T}{T}\right)_{lm} = \frac{4\pi}{3} \sum_{npq} i^l j_l(kR_{rec}/a) Y^*_{lm}(\vec{k}/k),$$

$$k = |\vec{k}|, \quad j_l(v) = \sqrt{\frac{\pi}{2v}} J_{l+1/2}(v). \tag{5}$$

Total multipole amplitudes $(\Delta T/T)_l$ follow from the expression

$$\left(\frac{\Delta T}{T}\right)^2_l \equiv \frac{1}{4\pi} \sum_{m=-l}^{l} \left(\frac{\Delta T}{T}\right)^2_{lm} =$$

$$= \frac{2l+1}{9} \sum_{\vec{k}} \sum_{\vec{k}'} \Phi(\vec{k})\Phi(\vec{k}') j_l(kR_{rec}/a) j_l(k'R_{rec}/a) P_l\left(\frac{\vec{k}\vec{k}'}{kk'}\right). \tag{6}$$

[1)] Results of this paper were presented in the author's plenary talk at the Texas/PASCOS'92 conference (Berkeley, USA, Dec. 13-18, 1992). After that the author became aware of recent papers [10] where similar results for the $L_1 = L_2 = L_3$ case were obtained independently.

Let $aL_1 \ll R_{hor}$ (a small 3-torus Universe). Then $\left(\frac{\Delta T}{T}\right)_l^2 \propto (2l+1)$ (this result was already obtained in [12]). In particular, for a stochastic isotropic δ-correlated spectrum of perturbations $\langle\Phi(\vec{k})\Phi(\vec{k}')\rangle = \Phi^2(k)\delta_{\vec{k}\vec{k}'}$, Eq.(6) takes the form:

$$\left(\frac{\Delta T}{T}\right)_l^2 = \frac{2l+1}{9}\sum_{\vec{k}} \Phi^2(k) j_l^2(kR_{rec}/a) \approx \frac{2l+1}{18}\sum_{\vec{k}} \Phi^2(k)\left(\frac{a}{kR_{rec}}\right)^2. \tag{7}$$

Therefore, amplitudes of low multipoles are much smaller (at least, in aL_1/R_{hor} times) than $\Delta T/T$ fluctuations at angles $\vartheta \sim aL_1/R_{hor}$ (in radians) [2)].

Multipole dependence of the observed large-scale $\Delta T/T$ fluctuations [9] does not follow this relation. Actually, it is much better fitted by the law $(\Delta T/T)_l^2 \propto \frac{2l+1}{l(l+1)}$ following from the inflationary scenario [13]. Thus, a sufficiently small 3-torus Universe is excluded by the data. Let us find more exact restrictions on the model. Another possible fit to the angular correlation function $\xi_T(\vartheta)$ of the COBE data is a Gaussian with the correlation angle $\vartheta_c = 13.5 \pm 2.5°$ (see [14]). This correspond to $l_m = 3-5$ where l_m is the multipole with the largest amplitude $(\Delta T/T)_l$. So, we may assume that $l_m \leq 6$.

First, consider modes with $npq \neq 0$ having a generic dependence on $\vec{r}$ and (θ,φ). For them, $k > 2\pi/L_3$. The quantity $(2l+1)j_l^2(v)$ considered as a function of integer l has a maximum at $l = l_m \leq 6$ for $v < 8.5$. Thus, a lower limit on the minimal topological size of the Universe follows:

$$aL_3 > 0.75R_{hor} \approx 9000\,Mpc. \tag{8}$$

Second, from modes with only one of the numbers n,p,q equal to zero, the modes with $q=0$, $n^2+p^2 \neq 0$ are the most important. If the contribution to $\Delta T/T$ from these modes dominates at low l, then L_3 may be much less than R_{hor}/a and the estimate (8) refers to the intermediate size aL_2. However, in this case the large-scale $\Delta T/T$ pattern has a symmetry plane due to the absence of dependence of the corresponding part of $\Phi(\vec{r})$ on z, i.e.

$$\frac{\Delta T}{T}(\theta,\varphi) = \frac{\Delta T}{T}(\pi-\theta,\varphi) \tag{9}$$

in some system of angular coordinates. For $aL_2 \gg R_{hor}$, the observed part of the Universe has the effective spatial topology T^1.

Finally, from modes with two of the numbers n,p,q equal to zero, the modes with $p=q=0$, $n \neq 0$ are the most interesting. They may give the main contribution to $\Delta T/T$ at low l if both L_2 and L_3 are much less than R_{hor}/a. Then the lower limit (8) refers to the maximal size aL_1 only but the large-scale $\Delta T/T$ pattern has a symmetry axis:

$$\frac{\Delta T}{T}(\theta,\varphi) = \frac{\Delta T}{T}(\theta). \tag{10}$$

For $aL_1 \gg R_{hor}$, we come to the case of the effective T^2 spatial topology.

Therefore, three possibilities exist:

1) all sizes of a 3-torus satisfy the estimate (8);

[2)] Here, the author would like to rectify an incorrect statement that $(\Delta T/T)_2 = 0$ in the case $L_1 = L_2 = L_3$ that appeared in the beginning of Ref.6 (fortunately, it was not used in any way in the following body of the paper). The correct statement is that this quantity is small (much less than 10^{-5}) if $aL_1 \ll R_{hor}$.

2) the estimate (8) refers to the two larger sizes but the large-scale $\Delta T/T$ pattern has a symmetry plane (9);
3) the maximal size of the torus alone satisfies (8) and Eq.(10) is valid.
The $\Delta T/T$ map obtained in [9] is noisy, so it does not immediately provides the real picture of primordial fluctuations. Nevertheless, Eq.(10) imposes so much symmetry that the third case is probably excluded already. The symmetry imposed by Eq.(9) in the second case is less evident visually, so more accurate maps are required to exclude this case definitely. If this is achieved, we may be sure that all topological sizes are at least of the order of the present cosmological horizon - in accordance with the prediction of the inflationary scenario. $\Delta T/T$ fluctuations produced by primordial gravitational waves are similar to those produced by adiabatic perturbations (though they depend on values of gravitational waves at all times between t_{rec} and t_0), so the same conclusions are expected to be valid in that case, too.

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