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DUAL FLIP-FLAP: A NEW PARADIGM OF COLLECTIVE DYNAMICS

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We define a new non-local dynamics as alternating inversions of the duality of the spin configurations on an expanded space of order-disorder variables. Some components of the novel procedure of the duality inversion are known, but the principal "absolutely parallel" operation of the dual complementation is new. The fixing of the duality reduces our dynamics to the Swendsen-Wang cluster algorithm.

Sufficiently close to the second-order phase transition dynamical properties, like thermodynamical ones, can be described by a set of critical exponents. The character of singularities of the thermodynamic quantities in the transition point is determined by the structure of the Hamiltonian and the number of components of the order parameter [1]. The hypothesis of the scale invariance makes it possible to obtain some relations between the critical exponents but is not sufficient to completely define the dynamic exponents [1,2]. So, even in the 2D Ising model for which exact values of the overall set of the thermodynamic exponents are known, the value of the dynamic exponent [3] $z = 2.183 \pm 0.005$, determining the homogeneous relaxation time $\tau \sim |T - T_c|^{-z}$, is known only as a result of the numerical simulations with local dynamic algorithms. The fact that z is close to 2, with the value of the exponent $\nu = 1$ of the correlation length $\xi \sim |T - T_c|^{-\nu}$, ensues from the local character of the dynamics. Here, like in the problem of random walk, to cover the distance ξ it takes time, proportional to ξ^2.

Different mechanisms ensuring relaxation can lead to diversity in critical dynamic behaviour [2]. Near the critical point the correlation length becomes very large, and lattice spins cannot adequately describe relevant degrees of freedom. The non-universality of the critical dynamics has been demonstrated by Swendsen and Wang [4] (SW), who have proposed the relaxation mechanism of a non-local type. The SW dynamics, operating with spin clusters, yields the value of the critical dynamic exponent [4] $z \approx 0.35$. Clusters in the SW conjecture seem to be closer to the genuine collective degrees of freedom, correctly describing the spatial and time correlations. Yet, the possibility of even a faster relaxation is still open to discussion [5,6].

2D Ising model is equvalent to a model of free fermions [7] which are known as the proper variables for describing static properties, whereas SW clusters are good objects for fast dynamics. The introduction of fermions is connected with duality transformation. Then there arises a natural question about the relationship between clusters, dual symmetry and fermions.

In what follows we shall study implications of the dual symmetry in the cluster description, and vice versa, the role of randomly generated clusters in the duality transformation. The solution of this problem will make it possible to construct

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a system of collective degrees of freedom, describing both the thermodynamics and dynamics in the critical region. In the present work we propose a new principle of relaxation for the 2D Ising model and point to possible ways to find generalizations for other systems.

The 2D Ising model, defined by the partition function

$$Z(\beta) = \sum_{\{\sigma\}} \prod_{x,\alpha} e^{\beta J \sigma_x \sigma_{x+\alpha}} \tag{1}$$

can be mapped by the duality transformation on itself [8] but at a different temperature T^*, which monotonically decreases with increasing temperature T of the original model,

$$e^{-2\beta^* J} = \tanh\beta J \tag{2}$$

Here β, β^* are inverse temperatures of mutually dual spin systems $\{\sigma\}$ and $\{\mu\}$, where $\{x\}$ are the sites and $\{\alpha\}$ are the two basic vectors of the square lattice considered further on. The transformation to dual spin variables $\{\mu\}$ is realized after the summation over spins of the original lattice $\{\sigma\}$ is performed [9]. The following form of the exponent in the partition function (1)

$$e^{\beta J \sigma_x \sigma_{x+\alpha}} = \cosh\beta J(1 + \sigma_x \sigma_{x+\alpha} \tanh\beta J)$$

is employed.

In the transformation to non-interacting clusters the summation over spins of the original lattice is preceeded by a different representation [10] of the same exponent

$$e^{\beta J \sigma_x \sigma_{x+\alpha}} = e^{\beta J}(e^{-2\beta J} + (1 - e^{-2\beta J})\delta_{\sigma_x, \sigma_{x+\alpha}}).$$

The clusters occuring in the result, consist of bonds placed on links of the lattice with the probability

$$p_{x,\alpha} = 1 - e^{-\beta J(\sigma_x \sigma_{x+\alpha} + 1)} \tag{3}$$

To unify the duality transformation and non-interacting clusters it is sufficient to study implications of the dual symmetry of the mixed four-spin correlation function

$$Q = \langle \sigma_x \sigma_{x+\alpha} \mu_x \mu_{x+\alpha} \rangle_\sigma = \langle \sigma_x \sigma_{x+\alpha} e^{-2\beta J \sigma_x \sigma_{x+\alpha}} \rangle_\sigma$$

in its representation via parameters of the clusters. Here the neighbouring couples of spins are positioned on the ends of intersecting dual links. In this case the subscript σ means the procedure of averaging

$$\langle(\ldots)\rangle_\sigma = \frac{\sum_{\{\sigma\}} e^{-\beta H\{\sigma\}}(\ldots)}{\sum_{\{\sigma\}} e^{-\beta H\{\sigma\}}}$$

over the original spin lattice having the temperature β. For disorder operators here their representations [11,12]

$$\mu_x = \prod_{-\infty}^{x} e^{-2\beta J \sigma\sigma'} \tag{4}$$

via the product of exponents along the contour at the dual lattice are used.

Expanding the exponent in the four-spin neighbouring-cross-linked correlation function Q, one can express it via the pair correlation function $G = \langle\sigma_x\sigma_{x+\alpha}\rangle_\sigma$ of the original lattice spins

$$Q = G\cosh 2\beta J - \sinh 2\beta J$$

An analogous expression through the pair correlation function of the dual spins

$$F = \langle\mu_x\mu_{x+\alpha}\rangle_\sigma = \langle e^{-2\beta J\sigma_x\sigma_{x+\alpha}}\rangle_\sigma$$

looks especially simple if the dual temperature is introduced

$$Q = \sinh 2\beta^* J - F\cosh 2\beta^* J.$$

The respective expressions for the correlation functions, pertaining to the dual lattice

$$G^* = \langle\mu_x\mu_{x+\alpha}\rangle_\mu,$$
$$F^* = \langle\sigma_x\sigma_{x+\alpha}\rangle_\mu = \langle e^{-2\beta^* J\mu_x\mu_{x+\alpha}}\rangle_\mu,$$
$$Q^* = \langle\sigma_x\sigma_{x+\alpha}\mu_x\mu_{x+\alpha}\rangle_\mu = \langle\mu_x\mu_{x+\alpha}e^{-2\beta^* J\mu_x\mu_{x+\alpha}}\rangle_\mu.$$

are obtained through the permutation $\beta \to \beta^*$, $\beta^* \to \beta$. Taking into account the well-known consequences $G^* = F$ and $F^* = G$ of the dual symmetry [11], enabling one at dual temperatures to relate pair correlation functions of order and disorder variables, it is easy to show the anti-self-duality of the mixed four-spin correlator Q

$$Q^* = -Q. \tag{5}$$

To clarify the consequences the dual symmetry in terms of lattice links, introduce probabilities $w_\pm$ of the same (opposite) orientation of spins on neighbouring sites ($w_+ + w_- = 1$). The mixed four-spin correlator Q can be represented via the probabilities $w_\pm$

$$Q = e^{-2\beta J}w_+ - e^{2\beta J}w_-$$

From the anti-self-duality (5) of Q there follow relations between probabilities $w_\pm$ on dual links

$$w_+^* = \cosh\beta J(e^{-\beta J}w_+ + e^{\beta J}w_-)$$

$$w_-^* = \sinh\beta J(e^{-\beta J}w_+ - e^{\beta J}w_-)$$

Here note the substantial temperature dependence which can be cancelled in the cluster representation.

Let w_1 ($w_0 = 1 - w_1$) be the probability of the presence (absence) of the cluster bond on the link of the lattice $\{\sigma\}$. The relationship between the probabilities $w_{1,0}$ and $w_\pm$ ensues from (3)

$$w_1 = (1 - e^{-2\beta J})w_+$$

$$w_0 = e^{-2\beta J}w_+ + w_-$$

The consequence of the anti-self-duality (5) of the cross-linked correlator Q in this representation is a very simple relation between probabilities of finding a cluster bond on the intersecting links of the original and dual lattices

$$w_1 + w_1^* = 1, \qquad\qquad w_0 + w_0^* = 1. \tag{6}$$

Thus we get a symmetric description of clusters on the original and dual lattices. Here the temperature dependence reveals itself only implicitly. At low temperatures bonds are largely localized on the original lattice whereas at high temperatures they move to the dual lattice. Let us connect one cluster bond with each intersection of links of the original and dual lattices, this bond can be positioned only on one of these two links. This simple scheme automatically ensures the dual symmetry (6) of probabilities $w_{1,0}$. An unambiguous correspondence of clusters on the dual lattices can be naturally termed dual complementation.

The absence of an unambiguous correspondence between spin configurations of the original and dual lattices is quite natural. Like in the conventional Fourier transformation, configurations on the dual lattice are the result of the summation performed over all configurations on the original lattice. However it is possible, using the dual complementation, to establish a correlation between spin configurations on mutually dual lattices.

First, we should build a configuration of bonds for the given spin configuration by means of probabilities $p_{x,\alpha}$ (3). Then by the dual complementation construct a configuration of bonds on the dual lattice. Then we have to restore the spin configuration for the given bond configuration. For this purpose we can randomly choose the common orientation of spins of each cluster.

We call this construction the duality inversion of the spin configuration. Note its stochastic character. Here the unambiguous procedure of the dual complementation is "wrapped" on the both sides with a probabilistic correspondence between spin and bond configurations.

So there naturally springs up a non-local method of the renewal of spins, which can actually be described as alternating inversions of the duality of spin configurations on an expanded space of the order-disorder variables. The proper variables to describe static properties, fermions, are defined in the same space as products of these variables [11]. The arising non-local dynamics can be named Dual Flip-Flap (DFF). The overall cycle consists of two duality inversions. If to "cut" the DFF cycle into operations of the dual complementation, we shall obtain two SW algorithms.

The duality transformation plays the same role as the Fourier transformation. The argument in favour of the DFF dynamics is the success in the removal of the critical slowing down in the Fourier acceleration algorithm [13,14], which actually is alternating application of Monte Carlo methods in the coordinate and momentum spaces. Yet, it is clear that the only straightforward way to verify the absence of the critical slowing down is to perform simulations, which are now in progress.

The diversity of perfect and inhomogeneous systems, possessing the dual symmetry [9,15] is a natural area for DFF applications. In the context of the hardware implementation it is worth noting the "absolutely parallel" character of the dual complementation procedure. The possibility of solving other problems pertaining to cluster algorithms at the hardware implementation has been demonstrated by a new specialized processor [16,17] employing the Wolff one-cluster algorithm [18].

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