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RADIATIVE CORRECTIONS TO THE PION BETA-DECAY

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QED radiative corrections to the width of the pion β-decay are calculated choosing muon decay as a normalization process. The hadronic effects in pion–virtual photon interactions are taken into account using the vector meson dominance model. Uncertainties due to strong interactions as well as the contribution from a background process we found to be small. The pure pionic radiative correction $\delta_\pi = -1.5\%$ was obtained.

The pion β-decay was considered first in 1954 year paper of Ya.B.Zeldovich and observed at first 30 years ago [1]. Recently an interest to this rare decay (its branching ratio $Br \sim 10^{-8}$) has grown again. The aim of the experiments planned [2] is the precise measurement of the V_{ud} matrix element of the Cabbibo-Cobayashi-Maskawa (CKM) matrix. In terms of the sum of squares of the first row in the CKM the present experimental results, dealt with the neutron β-decay, are $2.2\times\sigma$ higher and the ones from the superallowed nuclear β-decay are $2.5\times\sigma$ lower [3] than the Standard Model (SM) prediction (for three generations of quarks) — unity. The discrepancy of the average values of V_{ud} extracted from those experiments exceeded 1%. Whereas the accuracy being better than 0.5% is necessary to solve for instance the question about possible existence of an additional b' quark (if $V_{ub'} \neq 0$). The pion beta decay process is theoretically more clean than the nuclear– and nucleon beta decays, where a series of additional factors is to be taken into account.To absorb large radiative corrections connected with the Fermi coupling constant we use the muon decay as a "normalization" process.

Radiative corrections (RC) to the pion beta decay width in SM were calculated by A. Sirlin [4]. As a normalization process the superallowed nuclear $0^+ \to 0^+$ transitions were considered. The pure pionic correction $\delta_\pi = +3.1\%$ was obtained. Besides the other choice of a normalization process the difference of this result with our one $\delta_\pi = -1.5\%$ could follow from the different approaches used to take into account strong interaction effects. The scale of distances where the strong interaction are essential is much larger than the one of weak interactions. Loop integrals containing the photon-pion vertex will converge at loop momenta of order of the ρ-meson mass — the characteristic scale of strong interactions. To describe the strong interaction effects due to a coupling of a virtual photon with momentum k with a pion we use the vector meson dominance model (VDM). That leads to the following substitution for a photon propagator:

$$\frac{1}{k^2} \to \frac{m_\rho^2}{k^2(m_\rho^2 - k^2)} = \frac{1}{k^2} + \frac{1}{m_\rho^2 - k^2}. \tag{1}$$

This substitution is equivalent to the following choice of the cut-off parameter in the old–fashioned weak interaction theory: $\Lambda = m_h = m_\rho$. Uncertainty due to the choice of a characteristic hadronic mass m_h in the value of radiative corrections will have the order $\frac{\alpha}{\pi}\ln\frac{m_\rho}{m_\phi} \le 1\%$.

In section 1 of this paper we calculate the contribution of a radiative pion beta decay. In section 2 some other one-loop radiative corrections are considered, we discuss some sources of uncertainties due to strong interactions and estimate the contribution of a background process. They were found to be small.

The width of the pion β-decay in the Born approximation has the form [5]

$$\Gamma_0 = \frac{G^2\Delta^5|V_{ud}|^2}{\pi^3}\left(1-\frac{\Delta}{2m_\pi}\right)^3 I(\mu), \tag{2}$$

$$I(\mu) = \int\limits_\mu^1 dx\, x^2(1-x)^2\beta = \frac{1}{30}\Big[\left(1-\frac{9}{2}\mu^2-4\mu^4\right)\sqrt{1-\mu^2}+$$
$$+\frac{15}{2}\mu^4\ln\frac{1+\sqrt{1-\mu^2}}{2}\Big], \qquad \beta=\sqrt{1-\frac{\mu^2}{x^2}}, \tag{3}$$
$$\mu=\frac{m_e}{\Delta}, \qquad \Delta = m_{\pi^+}-m_{\pi^0} \approx 4.59\ \text{MeV}.$$

Large electroweak corrections arising from the W-boson self-energy, we note, are the same as for the muon decay width. The ratio of the π decay width and the muon decay one are free from the electroweak corrections:

$$\frac{\Gamma(\pi^+\to\pi^0 e^+\nu_e)}{\Gamma(\mu^+\to\bar\nu_\mu e^+\nu_e)} = \frac{|V_{ud}|^2 192}{30}\left(\frac{\Delta}{m_\mu}\right)^5\left[1+\frac{\alpha}{2\pi}\left(\pi^2-\frac{25}{4}\right)\right]\times \tag{4}$$
$$\times(1+\delta_\pi)\left(1-\frac{3\Delta}{2m_\pi}-\frac{5m_e^2}{\Delta^2}\right).$$

In the expression for the muon width we omitted the contributions of box type diagrams with the W,Z bosons. Their contributions do not exceed 0.01% [6].

1. Radiative $\pi\beta$-Decay

Consider now QED radiative corrections to the width of the π^+ β-decay

$$\pi^+(p_1)\to\pi_0(p_0)+e^+(p_e)+\nu(p_\nu)+\gamma(k). \tag{5}$$

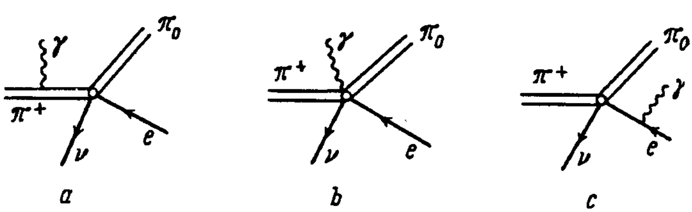

Fig.1. Feynman diagrams for the radiative π β-decay

The differential width of a radiative decay (Fig. 1) has the form

$$d\Gamma = -\frac{\alpha}{4\pi^2}\frac{d^3k}{\omega}d\Gamma_0(2\varepsilon_\nu\varepsilon_e-p_\nu p_e)^{-1}\times \tag{6}$$
$$\times\Big\{(2\varepsilon_\nu\varepsilon_e-p_\nu p_e)\Big[\frac{1}{\omega^2}+\frac{m_e^2}{(p_ek)^2}-\frac{2\varepsilon_e}{\omega(p_ek)}\Big]+$$
$$+(2\omega\varepsilon_\nu-kp_\nu)\Big[\frac{m_e^2}{(p_ek)^2}-\frac{\varepsilon_e+\omega}{\omega(p_ek)}\Big]+\frac{\varepsilon_\nu}{\omega}-\frac{2\varepsilon_\nu\varepsilon_e-p_ep_\nu}{(p_ek)}\Big\},$$

where $d\Gamma_0$ is the differential width in the Born approximation, $\varepsilon_e, \varepsilon_\nu, \omega$ are the energies of the electron, neutrino and photon respectively. We rearrange the phase space volume in (5) in the following way:

$$\int \frac{d^3k}{\omega}\frac{d^3p_e}{\varepsilon_e}\frac{d^3p_\nu}{\varepsilon_\nu}\frac{d^3p_0}{\varepsilon_2}\delta(p_1 - p_0 - p_e - p_\nu - k) = \tag{7}$$

$$= \frac{1}{m}\int p_e d\varepsilon_e \varepsilon_\nu d\varepsilon_\nu k d\omega 2(2\pi)^3 dc_\nu dc_\gamma \delta(\Delta - \varepsilon_e - \varepsilon_\nu - \omega) =$$

$$= 16\pi^3 m_\pi^{-1}\Delta^5 \int_\mu^1 dx \int_\nu^{1-x}(1 - x - z)dzxz\beta\beta_z \int_{-1}^1 dc_\nu \int_{-1}^1 dc_\gamma,$$

$$\beta = \sqrt{1 - \frac{\mu^2}{x^2}}, \quad \beta_z = \sqrt{1 - \frac{\nu^2}{z^2}}, \quad z = \frac{\omega}{\Delta}, \quad x = \frac{\varepsilon_e}{\Delta} \quad \nu = \frac{\lambda}{\Delta},$$

where λ is a "photon mass" parameter, c_ν, c_γ are the cosines of angles between the electron and the neutrino and the photon moments respectively. We omit in (7) the terms of order Δ/m_π compared with the terms of order unity. The corresponding error has the magnitude

$$\frac{\alpha}{\pi}\frac{\Delta}{m_\pi} \sim 10^{-4}. \tag{8}$$

Performing the angular integration over c_ν, c_γ and z one obtains

$$\int d\Gamma^{\pi^+\to\pi^0 e^+\nu\gamma} = \frac{G^2\Delta^5|V_{ud}|^2\alpha}{\pi^4}\int_\mu^1 dx\beta x^2(1-x)^2 \times \tag{9}$$

$$\times\Big\{\Big(-2 + \frac{1}{\beta}\ln\frac{1+\beta}{1-\beta}\Big)\Big[-1 + \ln(1-x) + \frac{1-x}{3x} +$$

$$+ \ln\frac{\Delta}{\lambda} + \frac{(1-x)^2}{24x^2}\Big] + 2(1 - \ln 2) + \frac{(1-x)^2}{12x^2} + \frac{1}{\beta}\int_0^\beta \frac{dt}{1-t^2}\ln\frac{\beta^2 - t^2}{\beta^2}\Big\}.$$

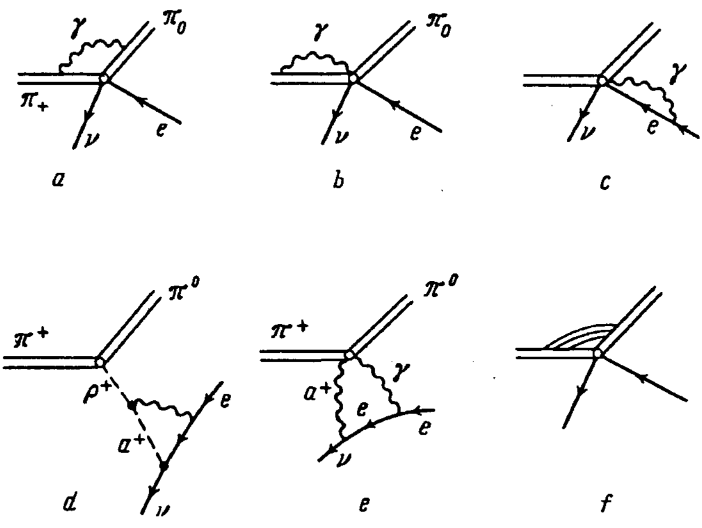

Fig.2. One-loop Feynman diagrams for the π β-decay

In order to calculate the correction due to a "virtual photon emission" we have to consider three one-loop Feynman diagrams (Fig.2). Two of them (Fig.2*b* and Fig.2*c*), which contain "contact" vertices, have the forms of a pion and positron self-energy loops. The doubled interference of their amplitudes with the Born one (summed over final spin states) has the form:

$$2\sum M_0^*(M_b^V + M_c^V) = 24m_\pi^2(2G|V_{ud}|)^2(-\frac{\alpha}{4\pi})\varepsilon_e\varepsilon_\nu \times$$
$$\times\left[\ln\frac{\Lambda^2}{m_\pi^2} + \frac{3}{2} + \frac{m_e^2}{\varepsilon_e m_\pi}(\ln\frac{\Lambda^2}{m_e^2} + \frac{3}{2})\right]. \quad (10)$$

The contributions from the terms in this equation which are proportional to m_e/m_π have an order of (8) and we will omit them further. The parameter Λ here is the momentum cut-off. Deriving (10) we omit terms of order

$$\frac{\alpha}{\pi}\frac{m_\pi^2}{\Lambda^2} < \frac{\alpha}{\pi}\frac{m_\pi^2}{m_\rho^2} < 10^{-4}, \quad (11)$$

they present the real magnitude of strong interaction's uncertainties. Applying the standard procedure of denominators joining and performing a loop momentum integration, an analogous to (10) expression for the first diagram could be obtained:

$$2\sum M_0^* M_a^V = 16m_\pi^2(2G|V_{ud}|)^2(-\frac{\alpha}{4\pi})\varepsilon_e\varepsilon_\nu \times \quad (12)$$
$$\times\left[-\frac{7}{2}\ln\frac{\Lambda^2}{m_\pi^2} - \frac{25}{6} + 2(\ln\frac{m_\pi^2}{m_e^2} - \frac{1}{\beta}\ln\frac{1+\beta}{1-\beta}) + \right.$$
$$\left. + \frac{2}{\beta}\ln\frac{1+\beta}{1-\beta}(\ln\frac{m_\pi^2}{\lambda^2} + \ln\frac{\beta^2\Delta^2 x^2}{m_\pi^2}) + \frac{4}{\beta}\int_0^\beta \frac{dt}{1-t^2}\ln\frac{1-t^2}{t^2}\right].$$

To consider the ultraviolet behavior of the amplitudes correctly, it is necessary to take into account the renormalization constants Z_e and Z_π for the positron and pion wave functions:

$$\frac{1}{\hat{p}-m} \to \frac{Z_e}{\hat{p}-m}, \qquad \frac{1}{p_1^2 - m_\pi^2} \to \frac{Z_\pi}{p_1^2 - m_\pi^2}, \quad (13)$$

where

$$Z_e = 1 + \frac{\alpha}{2\pi}(\ln\frac{m_e^2}{\lambda^2} - \frac{1}{2}\ln\frac{\Lambda^2}{m_e^2} - \frac{9}{4}), \quad (14)$$
$$Z_\pi = 1 + \frac{\alpha}{2\pi}(\ln\frac{\Lambda^2}{m_\pi^2} + \ln\frac{m_\pi^2}{\lambda^2} - \frac{3}{4}).$$

The total sum of the radiative π^+ β decay width which included the loop corrections does not contain the "photon mass" parameter λ:

$$\Gamma_0 + \Gamma^{(1)} = \Gamma_0(1+\delta_\pi),$$
$$\Gamma_0 = (G|V_{ud}|)^2(1 - \frac{3\Delta}{2m_\pi})\frac{\Delta^5}{\pi^3}I(\mu), \quad (15)$$

$$\delta_\pi = \frac{\alpha}{\pi}\left(\int_\mu^1 dx x^2(1-x)^2\beta\right)^{-1}\int_\mu^1 dx x^2(1-x)^2\beta \times \quad (16)$$

$$\times\left\{\frac{1}{\beta}\int_0^\beta \frac{dt}{1-t^2}\ln\frac{(\beta^2-t^2)t^2}{1-t^2}+\frac{1}{\beta}\Big(-\frac{1}{2}+\ln\frac{(1-x)}{\beta^2 x}+\right.$$
$$+\frac{(1-x)(1+7x)}{24x^2}\Big)\ln\frac{1+\beta}{1-\beta}-\frac{2(1-x)}{3x}+\frac{143}{48}-2\ln 2-$$
$$\left.-2\ln(1-x)+\ln\mu^2-\frac{1}{4}\ln\frac{m_\pi^2}{m_e^2}+\frac{3}{4}\ln\frac{\Lambda^2}{m_\pi^2}\right\}.$$

Numerical calculation give $\delta_\pi=-0.015$ for $\Lambda=m_\rho$.

2. Contribution of diagrams Fig.2d,e,f and background process

In paper [7] was argued that pion form factor can not remove all UV divergences. Namely diagrams Fig.2e,d were pointed out to have them. Diagram Fig.2d contains a vertex with a-meson, photon and ρ-meson. Such kind of vertices absent in a realistic chiral lagrangian theory with the vector current conservation. Really, the Wess-Zumino-Witten anomalous lagrangian L_{WZW} obtained by calibrated on the $SU(3)\cdot SU(3)$ group with the residual anomaly in the W. Bardeen form [8] does not contain any vertex without pseudoscalar mesons. As for the contribution of the diagram Fig.2e, it is described by the following term

$$L_{WZW}=\varepsilon^{\mu\nu\lambda\sigma}(d_\mu\pi^+)(d_\nu\pi^0)e^\lambda(a)e^\sigma(\gamma).$$

One can see that the contribution to δ_π of this diagram is UV finite and it has the order

$$\frac{\alpha}{\pi}\frac{m_+-m_0}{m_+}\leq 10^{-4}, \qquad (17)$$

which is less than the accuracy needed. Diagrams of the type Fig.2f contain the form factor F_π induced by strong interactions of the initial and final pions. It deviates from its static limit, unity, by terms being quadratic in the pions mass difference $F_\pi\sim 1+O((\frac{m_+-m_0}{m_+})^2)$.

As a possible background process to the pion beta decay one can consider the π_{e2} decay with two additional photons emitted:

$$\pi^+ \longrightarrow e^+ + \nu_e + \gamma + \gamma.$$

The process can imitate the final state of the pion beta decay with the subsequent decay of π_o into two photons. The contribution of the backgroun process was considered in details in [9]. For a reasonable deviation δm^2 of the squared two photon invariant mass m^2 from m_π^2 the corresponding contribution to δ_π would be small:

$$(\frac{\alpha m_e}{\Delta})^2(\ln\frac{\Delta}{m_e})^2\frac{\delta m^2}{m^2}\sim 10^{-4}\frac{\delta m^2}{m^2}. \qquad (18)$$

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