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Variational principle in canonical variables, Weber transformation and complete set of the local integrals of motion for dissipation–free magnetohydrodynamics

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The intriguing problem of the "missing" MHD integrals of motion is solved in the paper, i.e., analogs of the Ertel, helicity and vorticity invariants are obtained. The two latter were discussed earlier in the literature only for the specific cases, and Ertel invariant is first presented. The set of ideal MHD invariants obtained appears to be complete: to each hydrodynamic invariant corresponds its MHD generalization. These additional invariants are found by means of the fluid velocity decomposition based on its representation in terms of generalized potentials. This representation follows from the discussed variational principle in Hamiltonian (canonical) variables and it naturally decomposes the velocity field into the sum of "hydrodynamic" and "magnetic" parts. The "missing" local invariants are expressed in terms of the "hydrodynamic" part of the velocity and therefore depend on the (non–unique) velocity decomposition, i.e., they are gauge–dependent. Nevertheless, the corresponding conserved integral quantities can be made decomposition–independent by the appropriate choosing of the initial conditions for the generalized potentials. It is also shown that the Weber transformation of MHD equations (partially integration of the MHD equations) leads to the velocity representation coinciding with that following from the variational principle with constraints. The necessity of exploiting the complete form of the velocity representation in order to deal with general–type MHD flows (non–barotropic, rotational and with all possible types of breaks as well) in terms of single–valued potentials is also under discussion.
The new basic invariants found allows one to widen the set of the local invariants on the basis of the well–known recursion procedure.

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The fact that in the dissipation–free hydrodynamics there exist vorticity, helicity and Ertel invariants makes it evident that corresponding analogs have to exist for the ideal MHD as well. But in spite of the fact that for the dissipation–free MHD flows there exist additional topological invariants, namely, magnetic helicity and cross–helicity, introduced by Moffatt, [1], the analogs of the vorticity and helicity invariants have not been discussed with necessary completeness so far, cf., for instance, the recent review [2]. The related quantities were mentioned for the specific cases of symmetric flows in the works [3 – 5], the vorticity and helicity invariants for the incompressible flows were obtained in Refs. [6, 7]. The complete discussion of the generalized vorticity and helicity invariants for the compressible MHD flows without any restrictions relating to the flow symmetry, introducing of the generalized Ertel invariant will be presented below, see also [8]. Note here that these generalized invariants cannot be expressed in terms of the conventional Euler variables (the velocity, pressure, entropy, magnetic field, etc.) – their definition involves the specific decomposition of the velocity, which is natural in terms of the Weber transformation (or in terms of the Clebsch representation) and involves subsidiary fields. Suppose, this fact was the main obstacle in their derivation. The necessary subsidiary fields are related to the specific symmetry of the MHD equations. This symmetry is naturally expressed in terms of the Lagrange variables labelling the fluid particles: the corresponding symmetry group consists in relabelling, cf. Ref. [2] and citations therein. In the Hamiltonian approach these Lagrange labels are generalized coordinates and the velocity field is expressed in terms of their gradients multiplied by conjugate momenta constituting Clebsch–type representation, cf. [9] and [10, 2].

For the MHD case the velocity representation includes an additional term proportional to the magnetic field originally introduced in [11]. Below we use such representation that decomposes the velocity field into two parts, say, "hydrodynamic" one, which coincides with the velocity representation for the hydrodynamics, and the "magnetic" part (the latter vanishes identically for zero magnetic field). The generalized integrals of motion is then expressed using the hydrodynamic part of

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the velocity representation. Such decomposition of the velocity is not unique and the corresponding integrals of motion undergo some changes caused by the change in the generalized potentials (gauge transformations). We show that such changes can be eliminated by an appropriate choice of the initial conditions for the generalized potentials.

As it may seem, the velocity representation following from the variational principle is somewhat artificial, we prove its equivalence to that obtained from the generalization of the Weber transformation, cf. Refs. [12, 13], for the compressible non–barotropic MHD flows (for the incompressible flows it was done in Ref. [6], for barotropic case see Ref. [14]).

Based on the additional invariants found it is possible to derive a wide set of new invariants resulting from the recursion relations, cf. Refs. [15, 16]), see also [2, 10]). These sets of invariants are briefly discussed in the paper as well.

Let us start with the MHD equations in the Hamiltonian form. The dissipation–free MHD flows can be described in terms of the canonical variables, cf., for instance, Ref. [2]. Here we need the more complete set of the canonical variables enabling one to describe general–type flows. The example of such approach, allowing to deal with discontinuous motion as well, is presented in Refs. [17, 8]. For our case the appropriate Hamiltonian density $\mathcal{H}$ can be chosen in the form

$$\mathcal{H}(\mathcal{P},\mathcal{Q},\nabla\mathcal{Q}) = \frac{\rho\mathbf{v}^2}{2} + \rho\varepsilon + \frac{(\mathrm{curl}\mathbf{A})^2}{8\pi} + (\mathbf{M}\cdot\nabla\Lambda), \tag{1}$$

where ρ, ε and s denote density, internal energy and entropy, respectively, $\mathbf{A}$ is the vector potential, $\mathbf{H} = \mathrm{curl}\mathbf{A}$.[2)] The velocity field $\mathbf{v}$ is expressed in terms of the generalized coordinates, $\mathcal{Q} = (Q, \mathbf{M})$, and conjugate momenta, $\mathcal{P} = (P, \mathbf{A})$,

$$Q = (\varphi, \boldsymbol{\mu}, s), \quad P = (\rho, \boldsymbol{\lambda}, \sigma), \tag{2}$$

$$\begin{gathered}\mathbf{v} = \mathbf{v}(\mathcal{P},\mathcal{Q}) \equiv \mathbf{v}_h + \mathbf{v}_M\,, \\ \mathbf{v}_h = -\rho^{-1}P\nabla Q, \quad \mathbf{v}_M = -\rho^{-1}\left[\mathbf{H}\times\mathbf{M}\right].\end{gathered} \tag{3}$$

Here subindexes h and M correspond to the "hydrodynamic" and "magnetic" parts of the velocity field. The hydrodynamic part, $\mathbf{v}_h$, corresponds to the generalized Clebsch representation for the conventional hydrodynamics (compare, for instance, papers [17 – 19]), and the magnetic part, $\mathbf{v}_M$, coincides with the conventional MHD term if we replace the divergence–free field $\mathbf{M}$ by curl$\mathbf{S}$. The latter was first introduced by Zakharov and Kuznetsov, cf. Ref. [11].

The set of canonical equations

$$\dot{\mathcal{Q}} = \delta\mathcal{H}/\delta\mathcal{P}, \qquad \dot{\mathcal{P}} = -\delta\mathcal{H}/\delta\mathcal{Q}, \tag{4}$$

or, in explicit form

$$\dot\rho + \mathrm{div}(\rho\mathbf{v}) = 0, \quad D\varphi = w - v^2/2, \tag{5}$$

$$D\boldsymbol{\mu} = 0, \quad \dot\lambda_m + \mathrm{div}(\lambda_m\mathbf{v}) = 0, \tag{6}$$

$$Ds = 0, \quad \dot\sigma + \mathrm{div}(\sigma\mathbf{v}) = -\rho T, \tag{7}$$

$$\dot{\mathbf{A}} = [\mathbf{v}\times\mathrm{curl}\mathbf{A}] - \nabla\Lambda, \quad \dot{\mathbf{M}} = \frac{\mathrm{curl}\mathbf{H}}{4\pi} + \mathrm{curl}[\mathbf{v}\times\mathbf{M}], \tag{8}$$

includes mass conservation, entropy transport and magnetic field dynamic equations. Here w and T are the enthalpy and temperature, dot denotes time derivative, and $D \equiv \partial_t + (\mathbf{v}\cdot\nabla)$ is substantial (material) derivative.

From the velocity representation, Eq. (3), and Eqs. (5)–(8) it strictly follows that the velocity field satisfies Euler equation with the magnetic force taken into account,

$$\rho D\mathbf{v} = -\nabla p + (4\pi)^{-1}[\mathrm{curl}\mathbf{H}\times\mathbf{H}], \tag{9}$$

where p is the fluid pressure.

Note here that variation of the action

$$\mathcal{A} = \int dt \int d\mathbf{r}\,(\mathcal{P}\partial_t\mathcal{Q} - \mathcal{H}) \tag{10}$$

with respect to Λ leads to the divergence-free condition for the subsidiary field $\mathbf{M}$, div$\mathbf{M} = 0$. It is consistent with the equation of motion for $\mathbf{M}$, which leads to $\partial_t(\mathrm{div}\mathbf{M}) = 0$. Therefore, supposing that div$\mathbf{M} = 0$ holds for some initial moment we arrive at the conclusion that this is valid for the arbitrary moment. On the other hand, putting $\Lambda \equiv 0$ in the action leads to vanishing of the Λ term in the vector potential equation of motion, making it gauge–dependent. It proves convenient to deal with $\Lambda \neq 0$ that makes it possible to use different gauge conditions for the vector potential.

We do not include the subsidiary field Λ into the set of the canonical variables, but suppose it to be independent variable in the variational principle with the action defined by Eq. (10), dealing with the extended Hamiltonian description, cf. Ref. [20]. Alternatively, one can

[2)] The action differs slightly from that proposed in Ref. [17]. The main difference consists in introducing the vector potential for the magnetic field. Therefore, the canonical pair is $\mathbf{A},\mathbf{M}$ instead of $\mathbf{H},\mathbf{S}$, where $\mathbf{S} = \mathrm{curl}\mathbf{M}$. We do not deal with the discontinuous flows and omit the surface term in the action.

include it into the set of generalized coordinates. Denoting corresponding conjugate momentum π_Λ and adding to the Hamiltonian density, Eq. (1), the term $\pi_\Lambda \nu$ results in $\dot{\pi}_\Lambda = -\delta\mathcal{H}/\delta\Lambda = -\mathrm{div}\mathbf{M}$ $\dot{\Lambda} = \delta\mathcal{H}/\delta\pi_\Lambda = \nu$. The subsidiary function ν can be expressed in terms of other variables as $\nu = \partial_t \Delta^{-1}([\mathbf{v}\times\mathbf{H}] - \dot{\mathbf{A}}) + \nu_0$, where Δ denotes the Laplace operator and ν_0 is arbitrary solution of the Laplace equation. π_Λ is linear function of t, $\pi_\Lambda = \pi_\Lambda(t_0) - (t - t_0) div\mathbf{M}$, as it follows from (8). Putting $\pi_\Lambda(t_0)$ and $div\mathbf{M}(t) \equiv div\mathbf{M}(t_0) = 0$ we arrive at the specific case with the *a posteriori* zero–valued π_Λ. Note also that for $\mathbf{M}(t_0) = 0$ some generalized invariants become gauge–independent, see below.

The canonical description introduced for MHD is equivalent (obviously, up to the gauge transformations of the subsidiary fields) to the conventional description. This fact strictly follows from the generalization of the Weber transformation. The latter consists in partial integration of the Euler equation. Suppose that the fluid particles are labelled by Lagrange markers $\mathbf{a} = (a_1, a_2, a_3)$. Then the label of the particle passing through point $\mathbf{r} = (x_1, x_2, x_3)$ at time t is

$$\mathbf{a} = \mathbf{a}(\mathbf{r}, t), \quad \text{and, consequently,} \quad D\mathbf{a} = 0. \qquad (11)$$

The particle paths and velocities are given by the inverse function

$$\mathbf{r} = \mathbf{r}(\mathbf{a}, t), \quad \mathbf{v} = D\mathbf{r}(\mathbf{a}, t) = \partial\mathbf{r}/\partial t|_{\mathbf{a}=\mathrm{const}}. \qquad (12)$$

Representing the equation of motion in the form

$$Dv_k = -\frac{\partial w}{\partial x_k} + T\frac{\partial s}{\partial x_k} + [\mathbf{J}\times\mathbf{h}]_k, \ \mathbf{h} \equiv \frac{\mathbf{H}}{\rho}, \ \mathbf{J} \equiv \frac{\mathrm{curl}\mathbf{H}}{4\pi}, \qquad (13)$$

multiplying it by $\partial x_k/\partial a_i$ and performing simple transformations we have

$$D\left(v_k \frac{\partial x_k}{\partial a_i}\right) = \frac{\partial}{\partial a_i}(v^2/2 - w) + T\frac{\partial s}{\partial a_i} + [\mathbf{J}\times\mathbf{h}]_k \frac{\partial x_k}{\partial a_i}. \qquad (14)$$

Introducing subsidiary fields φ, σ and $\mathbf{M}$ governed by equations (compare second equations in (5), (7), (8))

$$D\varphi = w - \frac{v^2}{2}, \ D\left(\frac{\sigma}{\rho}\right) = -T, \ \dot{\mathbf{M}} = \mathrm{curl}\,[\mathbf{v}\times\mathbf{M}] + \mathbf{J} \qquad (15)$$

we can represent the r.h.s. of Eq. (14) as the sum of the substantial derivatives in accordance with identities:

$$T\frac{\partial s}{\partial a_i} = -D\left(\frac{\partial s}{\partial a_i}\frac{\sigma}{\rho}\right),$$

$$\frac{\partial}{\partial a_i}\left(w - \frac{v^2}{2}\right) = D\left(\frac{\partial\varphi}{\partial a_i}\right),$$

$$[\mathbf{J}\times\mathbf{h}]_k \partial x_k/\partial a_i = D\left([\mathbf{M}\times\mathbf{h}]_k \partial x_k/\partial a_i\right)$$

that makes it possible integration of the Euler equation (14):

$$-v_k\frac{\partial x_k}{\partial a_i} = \frac{\partial\varphi}{\partial a_i} + \frac{\partial s}{\partial a_i}\frac{\sigma}{\rho} + [\mathbf{h}\times\mathbf{M}]_k\frac{\partial x_k}{\partial a_i} + b_i, \ D\mathbf{b} = 0. \qquad (16)$$

Multiplying this relation by $\partial a_i/\partial x_j$ allows one to revert from Lagrangian, $(\mathbf{a}, t)$, to the Eulerian, $(\mathbf{r}, t)$, variables,

$$\mathbf{v} = -\nabla\varphi - b_k\nabla a_k - \frac{\sigma}{\rho}\nabla s - [\mathbf{h}\times\mathbf{M}]. \qquad (17)$$

The velocity representation following from the Weber transformation obviously coincides with the above introduced generalized Clebsch representation if one identifies $\mathbf{b}$ with $\boldsymbol{\lambda}/\rho$ and $\mathbf{a}$ with $\boldsymbol{\mu}$. This proves the above–mentioned equivalence between description of the general–type magnetohydrodynamic flows in terms of canonical variables introduced and the conventional description in Lagrange or Euler variables. Moreover, this indicates that the velocity representation of the form of Eq. (3) is the basic one. The possible reductions based upon the Darboux (in other terms – Pfaff's) theorem lead to the restrictions for the motion, or, otherwise, involve non single–valued functions, cf. Ref. [2].

The decomposition of the velocity in the "hydrodynamic" and "magnetic" parts allows one to obtain "missing" MHD invariants. Namely, strict but rather cumbersome calculations show that the generalized vorticity

$$\boldsymbol{\omega}_h \equiv \mathrm{curl}\mathbf{v}_h \qquad (18)$$

satisfy equation

$$\dot{\boldsymbol{\omega}}_h = [\nabla\rho\times\nabla p]/\rho^2 + \mathrm{curl}[\mathbf{v}\times\boldsymbol{\omega}_h]. \qquad (19)$$

For the barotropic flows $\boldsymbol{\omega}_h/\rho$ becomes frozen–in field,

$$D\left(\frac{\boldsymbol{\omega}_h}{\rho}\right) = \left(\frac{\boldsymbol{\omega}_h}{\rho}\cdot\nabla\right)\mathbf{v} \quad \text{for} \quad p = p(\rho), \qquad (20)$$

presenting clear generalization of the hydrodynamic vorticity for the MHD case. Note, that the quantity $\boldsymbol{\omega}/\rho$, $\boldsymbol{\omega} \equiv \mathrm{curl}\mathbf{v}$, is not frozen field due to the non potential character of the Lorentz force.

Starting with Eq. (19) one can prove the generalization of the Thompson theorem: circulation Γ of $\mathbf{v}_h$ over the closed substantial contour $\mathcal{C}$ lying on the entropy–constant surface is integral of motion,

$$D\Gamma \equiv D\oint_{\mathcal{C}}(\mathbf{v}_h\cdot d\mathbf{l}) = 0 \quad \text{for} \quad s|_{\mathcal{C}} = \mathrm{const}. \qquad (21)$$

Analogously, the pseudoscalar $h_H \equiv (\mathbf{v}_h \cdot \boldsymbol{\omega}_h)$ (generalized helicity) is conserved for the barotropic flows

$$\dot{h}_H + \mathrm{div}\mathbf{q}_H = 0, \quad h_H = (\mathbf{v}_h \cdot \boldsymbol{\omega}_h),$$
$$\mathbf{q}_H = h_H \mathbf{v} + (\chi - v^2/2)\boldsymbol{\omega}_h, \quad \chi \equiv \int dp/\rho. \tag{22}$$

The corresponding integral conserved quantity $\mathcal{I}_H$ can be obtained by integration of h_H over substantial volume $\widetilde{V}$ such that on its boundary the normal component of the generalized vorticity vanishes, $\omega_{hn} = 0$ (note that it is sufficient to require fulfillment of the latter condition for the initial moment only due to the frozen–in character of $\boldsymbol{\omega}_h/\rho$),

$$D\mathcal{I}_H = 0 \ \text{ for } \ (\boldsymbol{\omega}_h \cdot \mathbf{n})|_{\partial\widetilde{V}} = 0, \quad \mathcal{I}_H \equiv \int_{\widetilde{V}} d\mathbf{r} h_H. \tag{23}$$

For the incompressible case the generalized vorticity and helicity were introduced earlier in the paper [6].

The generalization of the Ertel invariant is represented by the quantity $\alpha_E = (\boldsymbol{\omega}_h \cdot \nabla s)/\rho$ following from the conventional hydrodynamics expression by substitution $\boldsymbol{\omega}_h = \mathrm{curl}\mathbf{v}_h$ instead of $\boldsymbol{\omega} = \mathrm{curl}\mathbf{v}$,

$$D\alpha_E = 0, \tag{24}$$

without any restrictions concerning the flow. Integration of $h_E = \rho\alpha_E$ over the substantial volume yields the following conserved integral

$$\mathcal{I}_E = \int_{\widetilde{V}} d\mathbf{r} h_E, \quad D\mathcal{I}_E = 0. \tag{25}$$

$\mathcal{I}_E$ in its structure is not gauge–invariant in contrast to the hydrodynamic case. Let us examine its change under gauge transformation $\mathbf{v}_h \Rightarrow \mathbf{v}'_h$, $\mathbf{v}_M \Rightarrow \mathbf{v}'_M$ with $\mathbf{v}'_h + \mathbf{v}'_M = \mathbf{v}_h + \mathbf{v}_M$. Then

$$\mathcal{I}'_E - \mathcal{I}_E = -\int_{\partial\widetilde{V}} d\Sigma(\mathbf{n} \cdot [\nabla s \times (\mathbf{v}'_M - \mathbf{v}_M)]). \tag{26}$$

Now we can proceed in the two ways. First, making use of identity $[\nabla s \times \mathbf{X}] = \mathrm{curl}(s\mathbf{X}) - s \cdot \mathrm{curl}\mathbf{X}$ and taking into account that integral of the first term vanishes (that is trivial for a closed boundary $\partial\widetilde{V}$ and assumes necessary decreasing of the integrand for the infinite volume $\widetilde{V}$), we have

$$\mathcal{I}'_E - \mathcal{I}_E = \int_{\partial\widetilde{V}} d\Sigma \Big(\mathbf{n} \cdot (\boldsymbol{\omega}'_M - \boldsymbol{\omega}_M)\Big) s. \tag{27}$$

From representation (26) immediately follows that integral Ertel invariant becomes gauge–independent for the substantial volume $\widetilde{V}$ chosen in such a way that its boundary coincides with the entropy–constant surface, $s|_{\partial\widetilde{V}} = \mathrm{const}$.

The second way is as follows. Substituting into Eq. (26) $\mathbf{v}'_M - \mathbf{v}_M = -[\mathbf{h} \times (\mathbf{M}' - \mathbf{M})]$ we obtain

$$\mathcal{I}'_E - \mathcal{I}_E = \int_{\partial\widetilde{V}} d\Sigma\, (\mathbf{n} \cdot [\nabla s \times [\mathbf{h} \times (\mathbf{M}' - \mathbf{M})]]). \tag{28}$$

Inasmuch as both $\mathbf{M}'$ and $\mathbf{M}$ satisfy the second equation in (8), their difference, $\overline{\mathbf{M}} = \mathbf{M}' - \mathbf{M}$, is governed by the corresponding homogeneous equation $\partial_t\overline{\mathbf{M}} = \mathrm{curl}[\mathbf{v} \times \overline{\mathbf{M}}]$, i.e. $\overline{\mathbf{m}} = \overline{\mathbf{M}}/\rho$ is frozen-in field. From this property follows that the vector quantity $\mathbf{W} \equiv [\nabla s \times [\mathbf{h} \times \overline{\mathbf{m}}]]$ entering the integrand is frozen–in field (it can be proved by strict calculations and also follows from the recursion relations between different type invariants, see, for instance, Refs. [2, 10] and Eqs. (31), (32) below). Therefore, if we restrict ourselves to zero initial condition for the field $\mathbf{M}$, $\mathbf{M}_{t=t_0} = 0$, the initial value of $\mathbf{W}$ becomes zero, and, consequently, initial value of its normal component on the surface of the substantial volume $\partial\widetilde{V}$ is zero. But the normal component of the frozen–in field is invariant of the motion. Thus we arrive at the conclusion that condition $(\mathbf{n} \cdot \mathbf{W}) = 0$ holds true for arbitrary moment, and $\mathcal{I}_E$ becomes gauge–invariant if we imply zero initial conditions for the subsidiary field $\mathbf{M}$. Note, that this choice do not impose any constraint on the motion.

There are also other possibilities to make $\mathcal{I}_E$ gauge–independent. For instance, we can restrict ourselves by such subset of the initial conditions for $\mathbf{M}$ that $\mathbf{M}|_{t=t_0} = f\mathbf{H}|_{t=t_0}$, where f is arbitrary function. (Then $\mathrm{div}\mathbf{M} = \mathrm{div}\mathbf{M}|_{t=t_0} = (\nabla f \cdot \mathbf{H}|_{t=t_0})$ and for the particular choice of f such one that $(\mathbf{H}|_{t=t_0} \cdot \nabla f) = 0$ we have $\mathrm{div}\mathbf{M} = 0$.) For these initial conditions $\mathbf{M}$ along with $\mathbf{M}'$ are collinear to $\mathbf{H}$ at the initial moment and therefore the initial value of $(\mathbf{n} \cdot \mathbf{W})$ is zero, and, consequently, it is zero for all moments. Thus we can make the conclusion that gauge dependence of the Ertel's invariant can be partly eliminated by appropriate choice of the initial conditions or substantial volumes.

The performed analysis of the additional MHD integrals of motion makes it quite evident that along with the well–known MHD integrals of motion, cf. [2], they form the basis of the MHD local invariants. Starting with these invariants one can obtain a wide set of invariants by means of recursion relations between different type invariants. Recall that for the hydrodynamic–type systems there exist four type of the local invariants. By definition they obey the following equations

$$D\alpha = 0, \quad D\mathbf{I} = 0, \quad D\mathbf{J} - (\mathbf{J} \cdot \nabla)\mathbf{v} = 0, \tag{29}$$

$$D\mathbf{L} + (\mathbf{L} \cdot \nabla)\mathbf{v} + [\mathbf{L} \times \mathrm{curl}\mathbf{v}] = 0. \tag{30}$$

Here α and $\mathbf{I}$ present the scalar and vector Lagrange invariants, $\mathbf{J}$ is frozen–in field, and $\mathbf{L}$ presents Lamb type mometum invariant, cf. Ref. [16]. The main representative of the fourth type invariants is the fluid density ρ, all other ρ–type invariants can be obtained by multiplying the local Lagrange invariants by ρ. The recursion relations

$$\mathbf{L}' = \nabla\alpha, \quad \alpha' = (\mathbf{J}\cdot\mathbf{L}), \tag{31}$$

$$\mathbf{J}' = [\mathbf{L}\times\mathbf{L}']/\rho, \quad \mathbf{L}' = \rho[\mathbf{J}\times\mathbf{J}'] \tag{32}$$

allows one to construct new invariants in terms of the initial ones. The procedure is somewhat different for the general–type motion and barotropic (or isentropic) flows. Let us consider the general case. Then the basic set of invariants can be chosen as

$$s, \quad \alpha_E\,, \quad \mathbf{h} = \mathbf{H}/\rho. \tag{33}$$

Applying recursion relations we first obtain new $\mathbf{L}$–type invariants, $\mathbf{L}_s = \nabla s$, $\mathbf{L}_E = \nabla\alpha_E$. These invariants allows us to get new α–type,

$$\alpha_P = (\mathbf{h}\cdot\nabla)s, \quad \alpha_E^{(1)} = (\mathbf{h}\cdot\nabla)\alpha_E\,, \tag{34}$$

and $\mathbf{J}$–type, $\mathbf{J}' = [\nabla s\times\nabla\alpha_E]/\rho$, invariants. The first invariant in Eq. (34) was obtained in the paper [21]. It is evident that invariants α_P and α_E are members of the infinite sets of (monomial) invariants

$$\alpha_P^{(m)} = \mathcal{D}^m s, \quad \alpha_E^{(m)} = \mathcal{D}^m\alpha_E\,, \quad m = 0, 1, \ldots\,, \tag{35}$$

where $\mathcal{D} = (\mathbf{h}\cdot\nabla)$. The first set was discussed in Ref. [2], and the second is the new one together with the generalized Ertel invariant α_E. The more general set of the scalar invariants can be presented by expression $\alpha_f = f(\{\alpha_P^{(m)}\}, \{\alpha_E^{(m)}\})$, where f is an arbitrary function. Note that the set $\{\alpha_f\}$ is closed relative to the operation $\mathcal{D}$.

We can proceed further constructing new scalar invariants by applying operation $\mathcal{D}_1 = (\mathbf{J}'\cdot\nabla)$ to the obtained at the previous step scalar invariants and obtaining new $\mathbf{L}$– and $\mathbf{J}$– type invariants. The problems relating to the complete set of the local invariants, their gauge invariance and specific types of flows will be discussed further.

The results obtained can be summarized as follows. First, the variant of introducing the canonical description of the MHD flows by means of the variational principle is presented. It is shown that in order to describe general–type MHD flows it is necessary to use in the generalized Clebsch–type representation for the fluid velocity field the vector Clebsch variables (the Lagrange markers and conjugate momenta) along with the entropy term (compare papers [18, 19] describing hydrodynamic case) and the conventional magnetic term. This complete representation allows one to deal with general–type MHD flows, including all type of breaks, see Ref. [17]. Second, the generalization of the Weber transformation for the MHD flows is performed. Third, the equivalence between the velocity representations, following, respectively, from the Weber transformation, and that introduced by means of the variational principle is proved. Forth, the generalized Ertel invariant for MHD flows is found. Fifth, the generalized vorticity and helicity invariants for the compressible barotropic MHD flows are obtained. Sixth, the relations between the local and integral invariants are discussed along with the gauge dependence of the latter. Seventh, as a consequence of the completeness of the representation proposed we arrive at the correct limit transition from the MHD to the conventional hydrodynamic flows. The results obtained allow one to deal with the complicated MHD problems by means of the Hamiltonian variables. The use of such approach was demonstrated for the specific case of incompressible flows in the series of papers devoted to the nonlinear stability criteria.

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