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MULTIPARTICLE PRODUCTION AND STATISTICAL ANALOGIES

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The study of singularities and zeros of the generating functions of multiplicity distributions is advocated. Some hints from well known probability distributions and experimental data are given. The statistical mechanics analogies provoke to look for a signature of phase transitions. The program of further experimental studies of the singularities is formulated.

Multiplicity distributions in high energy collisions of various projectiles and targets possess qualitatively similar (but quantitatively different!) behaviour. That is why many fits by some well known probability distributions have been tried. The ever more sensitive characteristics such as the ratio of cumulant to factorial moments have been proposed [1] and have revealed new features of experimental data [2]. Their understanding asks for further experimental and theoretical studies. It is proposed here to pay more attention to the structure of singularities and zeros of generating functions of multiplicity distributions. It is especially appealing in view of possible statistical analogies [3–6].

Let us define the generating function $G(z)$ of the probability distribution P_n by the relation

$$G(z) = \sum_{n=0}^{\infty}(1+z)^n P_n. \tag{1}$$

In what follows, we often use also the function

$$\Phi(z) = \ln G(z). \tag{2}$$

The (normalized) factorial (F_q) and cumulant (K_q) moments of the distribution P_n are related to them by the formulae

$$G(z) = \sum_{q=0}^{\infty}\frac{z^q}{q!}\langle n\rangle^q F_q \qquad (F_0 = F_1 = 1), \tag{3}$$

$$\Phi(z) = \sum_{q=1}^{\infty}\frac{z^q}{q!}\langle n\rangle^q K_q \qquad (K_1 = 1), \tag{4}$$

where $\langle n\rangle$ is the average multiplicity.

First we consider some distributions which provide analytical examples for the nature of the singularities. We start with the fixed multiplicity (FM) distribution when the sample of events of the same multiplicity (n_0) is chosen, then proceed to Poisson distribution (P) as a reference to independent emission processes and, finally, treat the gamma- (Γ), negative binomial (NB) and lognormal

(L) distributions widely used to fit experimental data at high energies. The corresponding functions $\Phi(z)$ look like

$$\Phi^{FM}(z) = n_0 \ln(1+z), \tag{5}$$

$$\Phi^{P}(z) = z\langle n\rangle, \tag{6}$$

$$\Phi^{\Gamma}(z) = -\mu \ln(1 - \frac{\langle n\rangle}{\mu}\ln(1+z)), \tag{7}$$

$$\Phi^{NB}(z) = -k\ln(1 - \frac{z\langle n\rangle}{k}), \tag{8}$$

where μ and k are the adjustable parameters. The lognormal distribution is here the only one which is not determined by its moments. From the integral representation of its generating function

$$\Phi^{L}(z) \to -\ln \int_0^\infty \exp[-\frac{(\ln x - \nu)^2}{2\sigma^2} + x\ln(1+z)]d(\ln x) \tag{9}$$

it is easily seen that its convergence radius is given by the inequality

$$|z+1|_L \leq 1, \tag{10}$$

i.e. the singularities come close to the point $z = 0$ but they are "soft" in the sense that the normalization condition $G(0) = 1$ persists. For other distributions the non-trivial (essential for our purposes) singularities are situated at

$$z_{NB} = k/\langle n\rangle, \tag{11}$$

$$z_{\Gamma} = \exp(\mu/\langle n\rangle) - 1, \tag{12}$$

$$z_P = \infty, \tag{13}$$

$$z_{FM} = -1. \tag{14}$$

Let us note that NB and Γ-singularities are close to $z = 0$ if the parameters k and μ are much less than $\langle n\rangle$. It is especially interesting because factorial and cumulant moments are calculated as q-th derivatives of $G(z)$ and $\Phi(z)$ at that point and the nearby singularity influences their behaviour substantially. In particular, it is important for the ratio of the moments

$$H_q = K_q/F_q, \tag{15}$$

which is identically equal to zero for Poisson distribution, alternates sign at each rank in case of fixed multiplicity, and is always positive for Γ and NB tending at asymptotically large ranks to zero as q^{-k} [7]. The different type of behaviour is predicted in QCD with strong decrease at low ranks followed by (quasi)oscillations at larger ranks [1, 7, 8]. It would be interesting to guess what singularity governs such a shape. There is no solution of the problem yet.

Let us get some guides from experimental data. In experiments with different projectiles and targets the adjustable parameters are different and energy dependent. Nevertheless, one can get qualitative estimate of the approximate locations of the singular points. In e^+e^--collisions, the NB-estimates give rise to $k/\langle n\rangle \sim 1$ (see, e.g., [9]) and, therefore, the singularity is situated at $z_{ee} \sim 1$, i.e. rather far from $z = 0$. It is much closer to the origin in hh-collisions where (see, e.g., [10]) $k/\langle n\rangle \sim 10^{-1}$. The AA-data is not so definite [11] (even though the lower statistics is slightly compensated by larger multiplicity) and give rise to $k/\langle n\rangle \leq 10^{-1}$ and, thus, to ever closer (to the origin) singularity. The singularities move to the origin with energy increase. Probably, these qualitative tendencies are related to somewhat similar regularities in the behaviour of the depth of the minimum of H_q found for various reactions (see [2, 11]) and to oscillations of H_q at large q (see below). Moreover, the oscillations of experimental distributions about the smooth NB-fit (see, e.g., [9]) could be connected with those oscillations. Their physical meaning could correspond to various number of subjets (ladders etc.) contributing at different multiplicities and should be checked in Monte-Carlo models. Another possible source of oscillations due to the cut-off of the multiplicity tail by conservation laws should die out asymptotically [12].

However, this cut-off plays an important role when one tries to restore the generating function directly from experimental data. Actually, the series (1) is replaced now by the partial sum in form of the polynomial in z

$$G_N(z) = \sum_{n=0}^{N}(1+z)^n P_n \tag{16}$$

with N equal to the highest observed multiplicity. Therefore the truncated generating function $G_N(z)$ has N complex conjugate zeros

$$G_N(z) = \prod_{j=1}^{N}\left(1 - \frac{z}{z_j}\right). \tag{17}$$

It was shown by DeWolf [13] that the zeros cover a circle in the complex z-plane for ee-events generated by JETSET Monte-Carlo program at 1000 GeV. It reminds of Lee-Yang zeros [3] in statistical mechanics. They seem to close in onto the singularity of $G(z)$ at some real $z = z_s > 0$ when N increases.

It is known [14] that the degree of infinity k of $G_N(z_s)$ is the same as the order of singularity of $G(z)$ at $z = z_s$ in case of algebraico-logarithmic behaviour and (for the algebraic singularity) is determined at $N \to \infty$ by the slope on double-log plot

$$\ln G_N(z_s) \to k \ln N + \ln(A_k/\Gamma(k+1)). \tag{18}$$

Here A_k is the residue of $G(z)$ at $z = z_s$ (NBD provides an example; see (8)).

Also, the order ρ of the integer function is given [15] by the formula

$$\lim_{n\to\infty} \frac{-\ln P_n(1+z_s)^n}{n \ln n} = \frac{1}{\rho}. \tag{19}$$

The above formulae can be used when interpreting experimental data.

The cumulants are determined [5, 13] by the moments of zeros locations

$$K_q = -\frac{(q-1)!}{\langle n\rangle^q}\sum_{j=1}^{N} z_j^{-q} = -\frac{(q-1)!}{\langle n\rangle^q}\sum_{j=1}^{N}\frac{\cos q\theta_j}{r_j^q}, \tag{20}$$

where we denote $z_j = r_j \exp(i\theta_j)$. Thus, the oscillations mentioned above are related to the phases of zeros.

The study of singularities of the generating function becomes more fruitful if one uses statistical mechanics analogies and recalls the Feynman fluid model [4–6]. The generating function is analogous to the partition function of the grand canonical ensemble and $\Phi(z)$ to free energy. The total rapidity range plays a role of the volume and the variable $1+z$ is just the fugacity. One can define the "pressure" $p(z)$ and the mean number of particles at given fugacity $\langle n(z)\rangle$ (proportional to the usual pressure and density) by the formulae

$$p(z) = \lim_{Y\to\infty} \frac{\Phi_N}{Y}, \tag{21}$$

$$\langle n(z)\rangle_N = (1+z)\frac{\partial\Phi_N}{\partial z}, \tag{22}$$

where $\Phi_N(z) = \ln G_N(z)$ and $\langle n(0)\rangle_N = \langle n\rangle$. Let us note that the behaviour of $\langle n(z)\rangle_N$ in the complex z-plane determined from experimental data should easily reveal zeros z_j of the function G_N [5] since it has poles exactly at the same loci z_j

$$\langle n(z)\rangle_N = \sum_{j=1}^{N} \frac{1+z}{z-z_j}. \tag{23}$$

The plots of $p(z)$ from experimental data about ee and hh-reactions extrapolated to $Y\to\infty$ have been shown in [6]. We have checked that the latest LEP data (e.g., [9]) well coincide with extrapolation used in [6] before the LEP data became available. The authors of [6] claim that there is no phase transition in ee-collisions. The qualitative conclusion from Figs.3*a* and 3*b* of [6] is that $p(z)$ increases at $z>0$ much faster in non-diffractive hh-collisions as compared to ee-collisions. It demonstrates that the hh-singularity is closer to the origin that corresponds to above conclusions. The increase would be even more drastic in case of AA-collisions (the data of EMU01 [11] were used for estimates) but it is strongly influenced by single events with very high multiplicity. Thus AA-analysis is hard to extend to large z. Probably, it has a physical origin since AA-collisions are the most suspected ones for phase transitions. Somewhat unexpected looks the constancy of $p(z)$ at $z<0$ for hh-collisions in Fig.3b of [6]. In statistical mechanics it would be a signature for phase transition. If supported by further studies, it would provide hints for theoretical speculations. Really, the problem of phase transition in systems with relatively small number of particles should be treated carefully. In particular, it depends on the steepness of increase of $p(z)$ with z. Some criteria of it are awaited for. However, the similarities may well happen to be mainly of formal nature and just the methods of analysis are comparable. Nevertheless, some physical models based on the analogy have been attempted [16–19].

Our preliminary qualitative results allow to formulate the further program of analysis of experimental data which consists of determining:

1) the radius of convergence of G_N (1) according to Cauchy ($P_n^{1/n}$) and D'Alembert (P_n/P_{n-1}) criteria;

2) the approach to the Carleman condition $\sum_{n=1}^{\infty} F_{2n}^{-1/2n} = \infty$ at high energies ($N\to\infty$);

3) location of zeros of $G_N(z)$ (formulae (17) or (23)) and their density;

4) the order of the singularity of $G(z)$ and its residue (18);

5) the order of the integer function (19);

6) the behaviour of the "pressure" $p(z)$ (21);

7) the behaviour of the "multiplicity" $\langle n(z) \rangle$ (22);

8) the higher derivatives of Φ_N (the fractional derivatives can be used also [20], especially, in connection with the classification of the phase transitions of non-integer order proposed recently [21]).

The extrapolations to $Y \to \infty$ should be attempted. It is quite probable that zeros locations will differ for different classes of processes (diffractive and non-diffractive; two- and three-jets etc). The drastic change in the behaviour of Φ_N or its derivatives must be carefully analysed to look for a possible signature of the phase transition. In parallel, the theoretical criteria of it in finite systems should be developed. We hope that the first stage of the program formulated above can provide some new insights into the physics of multiparticle production. More detailed results of it will be published elsewhere.

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