

# STATISTICAL PROPERTIES OF EIGENFUNCTIONS IN A DISORDERED METALLIC SAMPLE

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We calculate the distribution of eigenfunction amplitudes and the variance of the "inverse participation ratio" (IPR) in disordered metallic samples. The weak-localization corrections to the predictions of the Random Matrix Theory are found.

Statistical properties of disordered metals have attracted a considerable research interest last years. It was understood that the old problem of a quantum particle moving in a quenched random potential considered earlier in the context of Anderson localization and mesoscopic phenomena [1] exemplified a particular class of chaotic quantum systems and had much in common with such paradigmatic problems in the domain of Quantum Chaos as quantum billiards [2]. The Wigner-Dyson energy level statistics first found in the framework of random matrix theory (RMT) [3] and considered to be a "fingerprint" of quantum chaotic systems [4] was shown to be relevant for disordered metals as well [5,6]. This fact gave a boost to a broad application of RMT results for qualitative and quantitative description of mesoscopic conductors and stimulated a common interest to statistical characteristics of spectra of disordered systems [7].

At the same time less attention was paid to statistical properties of eigenfunctions in disordered or chaotic quantum systems. Recently, however, the distribution of eigenfunction amplitudes was shown to be relevant for description of fluctuations of tunnelling conductance across the "quantum dots" [8] as well as for understanding some properties of atomic spectra [9]. Besides, a so called "microwave cavity" technique has emerged [10] as a laboratory tool to simulate a disordered quantum system. This technique allows to observe directly eigenfunctions spatial fluctuations and was used in [11] to study experimentally the eigenfunction statistics in weak localization regime. All these facts make the issue of eigenfunction statistics to be of special interest and are calling for a detailed theoretical consideration.

In order to characterize eigenfunction statistics quantitatively, it is convenient to introduce a set of moments  $I_q = \int |\psi(r)|^{2q} d^d r$  of eigenfunction local intensity  $|\psi(r)|^2$  [12]. The second moment  $I_2$  is known as the inverse participation ratio (IPR). This quantity is a useful measure of eigenfunction localization: it is inversely proportional to a volume of a part of a system which contributes effectively to eigenstate normalization. For completely "ergodic" eigenfunctions covering randomly, but uniformly the whole sample  $I_2 \propto 1/V$ , with  $V$  being a system volume. If, in contrast, the eigenfunctions are localized, i.e. concentrated in a region of linear size  $\xi$ , the mean IPR scales as  $\overline{I_2} \propto \xi^{-D_2}$  where  $D_2$  is an effective dimension which can

be different from a spatial dimensionality  $d$  because of a multifractal structure of eigenfunctions [12]. Correspondingly, IPR fluctuations reflect *level-to-level* variations of eigenfunction spatial structure.

The most complete analytical study of statistical characteristics of eigenfunctions was performed for the cases of  $0d$  systems [13,14], as well as for strictly  $1d$  [15] and quasi  $1d$  [16,17] geometry. Some analytical results were obtained also for a system in the vicinity of localization transition in the dimensionality  $d = 2 + \epsilon$ ,  $\epsilon \ll 1$  [12] as well as for  $d \rightarrow \infty$  [18]. Let us note that in Ref. [13,14, 16–18] the supersymmetry method was used which is a very powerful tool to study distribution functions of various quantities characterizing eigenfunctions statistics.

In the present Letter we address systematically the issue of the eigenfunction statistics for arbitrary spatial dimensionality  $d$  in the weak localization domain. In the leading approximation (which neglects spatial structure of the system and treats it as a zero-dimensional one) these statistics are described by the RMT which predicts a Gaussian distribution of eigenfunctions amplitudes  $\psi(r)$  [13,17]. It is known since the paper by Altshuler and Shklovskii [6] that the diffusion motion of a particle in a metallic sample produces deviations of spectral statistics from what can be expected from RMT. To our best knowledge, the analogous problem for the eigenfunctions statistics in 2D and 3D systems has never been studied. It is just considered in the present paper. We use a recently developed method [19] which is based on the supersymmetry technics [5,20] and combines a perturbative elimination of fast diffusive modes (in spirit of renormalization group ideas) and consequent non-perturbative evaluation of a resulting  $0d$  integral. In this way, we calculate the deviations from the Gaussian distribution of  $\psi(r)$  in mesoscopic metallic samples. We calculate also the variance of the IPR which turns out to be of order of  $1/g^2$ , where  $g$  is the dimensionless (measured in units of  $e^2/h$ ) conductance of the sample.

In order to calculate the distribution of eigenfunction amplitude and to find the IPR variance we use the fact that relevant quantities can be expressed in terms of correlation functions of certain supermatrix  $\sigma$ -model [5,20]. A quite general exposition of the method can be found in [21] and is not repeated here. Depending on whether the time reversal and spin rotation symmetries are broken or not, one of three different  $\sigma$ -models is relevant, with orthogonal, unitary or symplectic symmetry group. We consider mostly the case of the unitary symmetry throughout the paper; for two other cases the calculations are similar, and only the results are presented.

The expressions for  $\overline{I_q}$  and  $\overline{I_2^2}$  (the bar standing for disorder averaging) in terms of the  $\sigma$ -model read as follows:

$$\overline{I_q} = \frac{-1}{2V} \lim_{\epsilon \rightarrow 0} \epsilon^{q-1} \frac{\partial}{\partial u} \bigg|_{u=0} \int DQ \exp \left\{ -\mathcal{F}^{(q)}(u, Q) + \frac{1}{t} \int d^d r \text{Str}(\nabla Q)^2 \right\}, \quad (1)$$

$$\overline{I_2^2} = \frac{1}{6V} \lim_{\epsilon \rightarrow 0} \epsilon^3 \frac{\partial^2}{\partial u^2} \bigg|_{u=0} \int DQ \exp \left\{ -\mathcal{F}^{(2)}(u, Q) + \frac{1}{t} \int d^d r \text{Str}(\nabla Q)^2 \right\}; \quad (2)$$

where

$$\begin{aligned} \mathcal{F}^{(q)}(u, Q) &= \int d^d r \{ \epsilon \text{Str}(\Lambda Q) + u \text{Str}^q(Q \Lambda k) \}, \\ Q &= T^{-1} \Lambda T, \quad \Lambda = \text{diag}(1, 1, -1, -1), \end{aligned}$$

$$k = \text{diag}(1, -1, 1, -1) ; \quad \frac{1}{t} = \pi \nu \mathcal{D} / 4 . \quad (3)$$

Here  $T$  is  $4 \times 4$  supermatrix belonging to the coset space  $U(1, 1 | 2)$ ,  $\mathcal{D}$  is the classical diffusion constant,  $\nu$  is the density of states and  $V$  is the system volume.

Generally speaking, the RMT predictions are applicable to a disordered metallic system under the following conditions:  $L \gg l$ ;  $E_c \gg \Delta$ , where  $L$  is the system size,  $l$  is mean free path,  $E_c = \hbar \mathcal{D} / L^2$  is the Thouless energy and  $\Delta$  is the mean level spacing. Eigenfunctions for such systems are known to be ergodic with amplitudes  $\psi(r)$  being uncorrelated (for  $|r - r'| \gtrsim l$ ) Gaussian distributed complex (real) variables for broken (unbroken) time-reversal symmetry respectively. This immediately gives [17,14]  $\overline{I_q^{(u)}} = q! / V^{q-1}$  and  $\overline{I_q^2} = \overline{I_q}^2$ , where the superscript  $u$  refers to the unitary symmetry.

In the framework of the  $\sigma$ -model formalism these results can be easily reproduced if one neglects any spatial variation of the supermatrix field  $Q(r)$ . Then eqs.(1), (2) are reduced to integrals over a single supermatrix which can be evaluated exactly. The corrections to RMT results have a form of a regular expansion in small parameter  $\Delta / E_c = g^{-1}$ . The systematic way to construct such an expansion can be briefly outlined as follows [19]. The matrix  $Q(r)$  is decomposed as  $Q(r) = T_0^{-1} \tilde{Q}(r) T_0$  where  $T_0$  is a spatially uniform matrix and  $\tilde{Q}$  describes all modes with non-zero momenta. When  $\Delta \ll E_c$ , the matrix  $\tilde{Q}$  fluctuates only weakly around the value  $\tilde{Q} = \Lambda$ . Thus, it can be expanded as  $\tilde{Q} = \Lambda \left( 1 + W + \frac{W^2}{2} + \dots \right)$  where  $W$  is a block off-diagonal supermatrix representing independent fluctuating degrees of freedom. Substituting this expansion into eqs.(1)–(3) and integrating out the “fast” modes one obtains an expression for renormalized functional  $\mathcal{F}_{eff}^{(q)}(u, Q_0)$ , where  $Q_0 = T_0^{-1} \Lambda T_0$  is an  $r$ -independent matrix (zero mode). The contribution of eliminated “fast” modes is expressed in terms of the diffusion propagator  $P(r_1, r_2)$ . For an isolated sample this propagator has the following form:

$$\begin{aligned} P(r_1, r_2) &= \sum_q \cos(qr_1) \cos(qr_2) P(q) , \\ P(q) &= \frac{1}{2\pi\nu V} \frac{1}{\mathcal{D}q^2 + \epsilon} , \quad q = \pi \left( \frac{n_1}{L_1}, \dots, \frac{n_d}{L_d} \right) , \\ n_i &= 0, \pm 1, \pm 2, \dots , \quad \sum_i n_i^2 > 0 , \end{aligned} \quad (4)$$

where the system is thought to be of the size  $L_1 \times L_2 \times \dots \times L_d$ . Finally, the integrals over  $Q_0$  are performed exactly.

Applying this method to eqs.(1), (3) one obtains:

$$\overline{I_q^{(u)}} = \frac{q!}{V^{q-1}} \left\{ 1 + \frac{a_1}{g} q(q-1) + O\left(\frac{1}{g^2}\right) \right\} , \quad (5)$$

where  $g = 2\pi\nu \mathcal{D} L^{d-2}$  is the conductance of the sample. The value of the coefficient  $a_1 = g \sum_q P(q)$  depends on the spatial dimension and is equal to  $a_1 = 1/6$  in quasi 1d systems. For  $d \geq 2$  the corresponding sum over momenta  $q$  diverges at large  $|q|$  and is to be cut off at  $|q| \sim l^{-1}$ . This gives  $a_1 = \frac{1}{2\pi} \ln L/l$  for  $d = 2$  and  $a_1 \propto L/l$  for  $d = 3$ .

Knowing all the moments  $\overline{I_q^{(u)}}$  it is easy task to restore the whole probability distribution  $\mathcal{P}(y)$  of the eigenfunction local intensity  $y = V|\psi(r)|^2$ :

$$\mathcal{P}^{(u)}(y) = e^{-y} \left[ 1 + \frac{a_1}{g} (2 - 4y + y^2) + O\left(\frac{1}{g^2}\right) \right]. \quad (6)$$

The corresponding equations for systems with unbroken time reversal symmetry (orthogonal and symplectic  $\sigma$ -model) are as follows [22]:

$$\mathcal{P}^{(o)}(y) = \frac{e^{-y/2}}{\sqrt{2\pi y}} \left[ 1 + \frac{a_1}{g} \left( \frac{3}{2} - 3y + \frac{y^2}{2} \right) + O\left(\frac{1}{g^2}\right) \right], \quad (7)$$

$$\mathcal{P}^{(sp)}(y) = 4ye^{-2y} \left[ 1 + \frac{a_1}{g} (3 - 6y + 2y^2) + O\left(\frac{1}{g^2}\right) \right]. \quad (8)$$

The leading terms here reproduce the well-known Porter-Thomas distribution which is the RMT result [3]; the rest is the weak localization correction. In the quasi 1d sample these expressions coincide with that obtained in [17] if one identifies the scaling parameter introduced in [17] as  $x = g^{-1}$ . Equations (6), (7), (8) are valid up to  $y \lesssim \sqrt{g/a_1}$ . For larger values of  $y$  (i.e. in the far "tail") the distribution function  $\mathcal{P}(y)$  differs strongly from that of the random matrix theory and can not be found by the method used here.

Very recently, the distribution of eigenfunctions amplitude,  $\mathcal{P}(\dagger)$ , was studied experimentally in a microwave cavity with disorder [11]. The reported results are in good agreement with our formula (7).

Now we turn to the consideration of IPR fluctuations. It turns out that IPR variance is of order of  $1/g^2$ . Thus the expression (5) is insufficient for our needs and should be extended to the next order. By the same method we find:

$$\begin{aligned} \overline{I_2^{(u)}} &= \frac{2}{V} \left[ 1 + \frac{2a_1}{g} + \frac{1}{g^2} (2a_1^2 - 5a_2) + O\left(\frac{1}{g^3}\right) \right], \\ \overline{\left[ I_2^{(u)} \right]^2} &= \left( \frac{2}{V} \right)^2 \left[ 1 + \frac{4a_1}{g} + \frac{8a_1^2}{g^2} - \frac{2a_2}{g^2} + O\left(\frac{1}{g^3}\right) \right]. \end{aligned}$$

Here the coefficient  $a_2$  is defined as  $a_2 = g^2 \sum_q P^2(q)$  and is equal to

$$a_2 = \frac{1}{\pi^4} \sum_{n_i \geq 0; n_d > 0} \frac{1}{(n_1^2 + \dots + n_d^2)^2}; \quad (9)$$

the sum being convergent for  $d < 4$ . For quasi 1d samples  $a_2 = 1/90$  and the found expressions coincide with the results of [16,17].

Thus, we find the following expression for the relative variance of the IPR distribution:

$$\delta^{(u)}(I_2) \equiv \frac{\overline{\left[ I_2^{(u)} \right]^2} - \left[ \overline{I_2^{(u)}} \right]^2}{\left[ \overline{I_2^{(u)}} \right]^2} = \frac{8a_2}{g^2} + O\left(\frac{1}{g^3}\right). \quad (10)$$

This result demonstrates that for a metallic sample the distribution function of IPR  $\mathcal{P}(I_2)$  has a form of a narrow peak with a typical width of order of  $\delta^{1/2} \propto g^{-1} \ll 1$ . When  $g \rightarrow \infty$   $\mathcal{P}(I_2) \rightarrow \delta(I_2 - \overline{I_2})$ , the result one expects from

RMT. For the orthogonal and symplectic symmetry cases we find  $\delta^{(o)}(I_2) = 32a_2/g^2$  and  $\delta^{(sp)}(I_2) = 2a_2/g^2$  respectively.

In conclusion, we have studied deviations of the eigenfunction statistical characteristics in a disordered metallic sample from those predicted within the Random Matrix Theory.

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  22. For all symmetry cases, the conductance  $g$  is defined as  $g = 2\pi\nu DL^{d-2}$ , without including the factor 2 due to the spin.