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Qubit decoherence by Gaussian low-frequency noise

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We have derived explicit non-perturbative expression for decoherence of quantum oscillations in a qubit by Gaussian low-frequency noise. Decoherence strength is controlled by the noise spectral density at zero frequency while the noise correlation time τ determines the time t of crossover from the $1/\sqrt{t}$ to the exponential suppression of coherence. We also performed Monte Carlo simulations of qubit dynamics with noise which agree with the analytical results.

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Despite the large number of successful demonstrations of coherent quantum oscillations in individual [1 – 8] and coupled [9] Josephson-junction qubits, quantitative understanding of these oscillations is so far limited. The main area of discrepancy between experiment and theory is qubit decoherence. The typical quality factors of reported oscillations, while not as large as required by potential applications in quantum computation, are still quite large in physics term (typically not less than $20 \div 30$). This fact should imply weak decoherence describable by the standard perturbation theory in qubit-environment coupling (see, e.g., [10]). Several basic features of this theory, however, do not agree with experimental observations. Most importantly, observed decay time T_2 of coherent oscillations is typically shorter than the energy relaxation time T_1 even at optimal qubit bias points [4, 3, 11] where perturbation theory predicts no pure dephasing terms. Another discrepancy is between the observed two-qubit decoherence rate [9] and its values that can be obtained from the perturbation theory under natural assumptions [12].

Qualitatively, the basic reason for discrepancy between T_1 and T_2 is the low-frequency noise that can reduce T_2 without changing significantly the relaxation rates. Mechanisms of low-frequency, or specifically $1/f$, noise exist in all solid-state qubits: background charge fluctuations for charge-based qubits [13], impurity spins or trapped fluxes for magnetic qubits [14]. Manifestations of this noise are observed in the echo-type experiments [11]. Low-frequency noise for qubits is also created by the electromagnetic fluctuations in filtered control lines.

The goal of our work is to develop quantitative theory of low-frequency decoherence by studying qubit dynamics under the influence of Gaussian noise with small characteristic amplitude v_0 and long correlation time τ. In this case, we obtained explicit non-perturbative expression describing decay in time of coherent qubit oscillations. The strength of decoherence in this expression is controlled by the noise spectral density at zero frequency, $S_v(0) \propto v_0^2\tau$. For long correlation times $\tau \gg \Delta^{-1}$, where Δ is the qubit tunnel amplitude, $v_0^2\tau$ can be large even for weak noise $v_0 \ll \Delta$ and our analytical results are exact as function of $v_0^2\tau$ in this limit. We also performed direct numerical simulations of the low-frequency qubit decoherence. The simulation results confirm analytical expressions and show that our main conclusions: cross-over from the $1/\sqrt{t}$ to the exponential suppression of coherence at time $t \simeq \tau$; and the strength of decoherence controlled by the noise spectral density $S_v(0)$ at zero frequency, are valid for quite large noise amplitudes v.

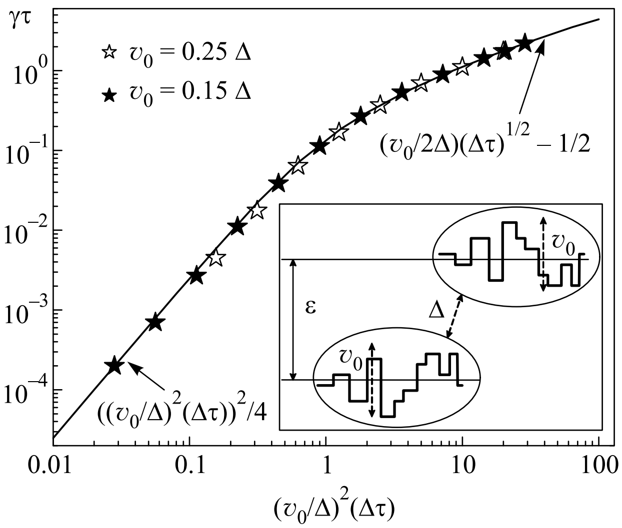
The Hamiltonian of a qubit with a fluctuating bias energy $v(t)$ (see inset in Figure) is:

$$H = -\frac{1}{2}[\Delta\sigma_x + (\varepsilon + v(t))\sigma_z]\,, \qquad (1)$$

where ε is the average bias, and σ's here and below denote Pauli matrices. In this work, we mostly focus on the situation when the noise $v(t)$ has characteristic correlation time τ, i.e., the noise correlation function and its spectral density can be taken as

$$\langle v(t)v(t')\rangle = v_0^2 e^{-|t-t'|/\tau}\,, \quad S_v(\omega) = \frac{2v_0^2\tau}{1+(\omega\tau)^2}\,, \quad (2)$$

where v_0 is the typical noise amplitude and $\langle ...\rangle$ denotes average over different realizations of noise. We assume that the temperature T of the noise-producing environment is large on the scale of the cut-off frequency $1/\tau$, and it can be treated as classical. In the regime of in-



The rate γ of exponential qubit decoherence at long times $t \gg \tau$ for $\varepsilon = 0$ and noise with characteristic amplitude v_0 and correlation time τ. Solid line gives analytical results from Eq. (8). Symbols show γ extracted from Monte Carlo simulations of qubit dynamics. Inset shows schematic diagram of qubit basis states fluctuating under the influence of noise $v(t)$

terest, $1/\tau \ll \Delta$, the temperature can obviously be still small on the qubit energy scale.

The two effects of the weak noise on the dynamics of the qubit (1) are the transitions between two energy eigenstates with energies $\pm\Omega/2$, $\Omega \equiv (\Delta^2 + \varepsilon^2)^{1/2}$, and "pure" (unrelated to transitions) dephasing that suppresses coherence between these states. Within the standard perturbation theory, the transition rate is proportional to $S_v(\Omega) = 2v_0^2/\Omega^2\tau$. One can see that the condition of weak noise $v_0 \ll \Delta$ makes the transition rate small compared both to Δ and $1/\tau$ ensuring that the perturbation theory is sufficient for the description of transitions. As discussed qualitatively in the introduction, the fact that the noise correlation time is long, $\tau \gg \Delta^{-1}$, makes the perturbation theory inadequate for the description of pure dephasing. For low-frequency noise, a proper (non-perturbative in $v_0^2\tau$) description is obtained by looking at the accumulation of the noise-induced phase between the two instantaneous energy eigenstates. If $v_0 \ll \Delta$, one can determine the rate of accumulation of this phase by expanding the energies in noise amplitude $v(t)$. Also, in this case the dephasing rate is larger than the transition rate and can be calculated disregarding the transitions. The factor $F(t)$ describing suppression in time of coherence between the two states (i.e., suppression of the off-diagonal element ρ_{12} of the qubit density matrix in the energy basis: $\rho_{12}(t) = F(t)\rho_{12}(0)e^{-i\Omega t}$) can be written then as follows:

$$F(t) = \langle \exp\{-i\int_0^t [\frac{\varepsilon v(t')}{\Omega} + \frac{\Delta^2 v^2(t')}{2\Omega^3}]dt'\}\rangle\,. \qquad (3)$$

For Gaussian noise, the correlation function (2) determines the noise statistics completely, and it is convenient to take the average in Eq. (3) by writing it as a functional integral over noise. For this purpose, and also for use in the numerical simulations, we start with the "transition" probability $p(v_1, v_2, \delta t)$[15] for the noise to have the value v_2 a time δt after it had the value v_1:

$$p(v_1, v_2, t) = [2\pi v_0^2(1-e^{-2\delta t/\tau})]^{-1/2} \exp\{-\frac{1}{2v_0^2}\frac{(v_2 - v_1 e^{-\delta t/\tau})^2}{1-e^{-2\delta t/\tau}}\}. (4)$$

Using this expression we introduce the probability of specific noise realization as $p_0(v_1) \cdot p(v_1, v_2, \delta t_1) \cdot p(v_2, v_3, \delta t_2) \cdot ...$, where $p_0(v) = (2\pi v_0^2)^{-1/2}\exp\{-v^2/2v_0^2\}$ is the stationary Gaussian probability distribution of v. Taking the limit $\delta t_j \to 0$ we see that the average over the noise can be written as the following function integral:

$$\langle ...\rangle = \int dv(0)dv(t)Dv(t')... \exp\left\{-\frac{v(0)^2 + v(t)^2}{4v_0^2} - \right.$$
$$\left. - \frac{1}{4v_0^2\tau}\int_0^t dt'(\tau^2\dot{v}^2 + v^2)\right\}. \qquad (5)$$

Since the average in Eq. (3) with the weight (5) is now given by the Gaussian integral, it can be calculated straightforwardly:

$$F(t) = F_0(t)\exp\left[-\alpha^2\left(\frac{\nu t}{\tau} - 2[\coth\frac{\nu t}{2\tau} + \nu]^{-1}\right)\right], \quad (6)$$
$$F_0(t) = e^{t/2\tau}[\cosh(\nu t/\tau) + \frac{1+\nu^2}{2\nu}\sinh(\nu t/\tau)]^{-1/2},$$

where $\nu \equiv \sqrt{1 + 2iv_0^2\Delta^2\tau/\Omega^3}$ and $\alpha \equiv \varepsilon\tau v_0/\Omega\nu^{3/2}$.

Equation (6) is our main analytical result. To analyze its implications, we start with the case $\varepsilon = 0$, where pure qubit dephasing vanishes in the standard perturbation theory. Dephasing (6) is still non-vanishing and its strength depends on the noise spectral density at zero frequency $S_v(0) = 2v_0^2\tau$ through $\nu = \sqrt{1+is}$, $s \equiv S_v(0)/\Delta$. For small and large times t Eq. (6) simplifies to:

$$F(t) = \begin{cases} \left[\dfrac{1+t/\tau}{1+t/\tau+ist/2\tau}\right]^{1/2}, & t \ll \tau, \\ 2\sqrt{\nu}e^{-(\gamma+i\delta)t}/(1+\nu)\,, & t \gg \tau, \end{cases} \qquad (7)$$

where

$$\gamma = \frac{1}{2\tau}\left[\left(\frac{(1+s^2)^{1/2}+1}{2}\right)^{1/2} - 1\right]. \qquad (8)$$

Besides suppressing coherence, the noise also shifts the frequency of qubit oscillations. The corresponding frequency renormalization is well defined for $t \gg \tau$:

$$\delta = \frac{1}{2\tau}\left[\frac{(1+s^2)^{1/2}-1}{2}\right]^{1/2}. \qquad (9)$$

Suppression of coherence (7) for $t \ll \tau$ can be qualitatively understood as the result of averaging over the static distribution of noise v. In contrast to this, at large times $t \gg \tau$, the noise appears to be δ-correlated, the fact that naturally leads to the exponential decay (7). This interpretation means that the two regimes of decay should be generic to different models of the low-frequency noise. Crossover between the two regimes takes place at $t \simeq \tau$, and the absolute value of $F(t)$ in the crossover region can be estimated as $(1+s^2)^{-1/4}$, i.e. s determines the amount of coherence left to decay exponentially. The rate (8) of exponential decay shows a transition from the quadratic to square-root behavior as a function of $S_v(0)$ that can be seen in Figure, which also shows the decay rate extracted from numerical simulations of Gaussian noise. Our numerical procedure was based on direct Monte Carlo simulations of coherent oscillations of a qubit with Hamiltonian (1) that start in one of the eigenstates of the σ_z operator. The qubit density matrix was averaged over up to 10^7 realizations of noise that were built using the transition probability (4). The rate γ of pure dephasing was extracted from the long-time behavior of the off-diagonal element of the density matrix by subtracting the transition-induced dephasing rate $S_v(\Delta)/4 = v_0^2/(2\Delta^2\tau)$ from the total oscillation decay rate. One can see from Figure that analytical and numerical results agree well for quite large noise amplitudes v.

Non-zero qubit bias ε leads to additional dephasing $F(t)/F_0(t)$ described by the last exponential factor in Eq. (6). The contribution from $F_0(t)$ is of the same form as in $\epsilon = 0$ case but now with $s \to s(\Delta/\Omega)^3$. Similarly to $F_0(t)$, the additional dephasing exhibits the crossover at $t \simeq \tau$ from "inhomogeneous broadening" (averaging over the static distribution of the noise v) to exponential decay at $t \gg \tau$. In contrast to the zero-bias case, the short-time decay is now Gaussian:

$$\ln\left[\frac{F(t)}{F_0(t)}\right] = -\frac{\varepsilon^2}{\Omega^2}\cdot\begin{cases} v_0^2 t^2/2, & t \ll \tau, \\ v_0^2 \tau t/(1+is(\Delta/\Omega)^3), & t \gg \tau. \end{cases}$$

We see that, again, the rate of exponential decay depends non-trivially on the noise spectral density $S_v(0)$, changing from direct to inverse proportionality to $S_v(0)$ at small and large s, respectively.

Our approach can be used to calculate the rate of exponential decay at large times t for Gaussian noise with arbitrary spectral density $S_v(\omega)$. Such a noise can be represented as a sum of noises (2) and appropriate transformation of variables in this sum enables one to write the average over the noise as a functional integral similar to (5). For calculation of the relaxation rate at large t, the boundary terms in the integral (5) can be neglected and it is dominated by the contribution from the "bulk" which can be conveniently written in terms of the Fourier components

$$v_n = (2/t)^{1/2}\int_0^t dt' v(t')\sin\omega_n t', \quad \omega_n = \pi n/t.$$

Then, $\langle ... \rangle = \int Dv ... \exp\{-(1/2)\sum_n |v_n|^2/S_v(\omega_n)\}$. Combining this equation and Eq. (3) we get at large t:

$$F(t) = \exp\left\{-\frac{t}{2}\left[\frac{\varepsilon^2\Omega S_v(0)}{\Omega^3 + iS_v(0)\Delta^2} + \frac{1}{\pi}\int_0^\infty d\omega \ln(1+iS_v(\omega)\Delta^2/\Omega^3)\right]\right\}. \qquad (10)$$

For unbiased qubit, $\varepsilon = 0$, this equation coincides with the one obtained by more involved diagrammatic perturbation theory in quadratic coupling [16].

In summary, we developed non-perturbative theory of qubit dephasing by Gaussian low-frequency noise and performed Monte Carlo simulations of qubit dynamics with this noises. The theory agrees well with simulations and shows that the decoherence strength is controlled by the noise spectral density at zero frequency. It allows for generalizations in several experimentally-relevant directions and should be useful for analysis of observed shapes of quantum qubit oscillations.

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