Pis'ma v ZhETF, vol. 80, iss. 4, pp. 247 – 254

Expansion in Feynman graphs as simplicial string theory

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Submitted 8 July 2004

We show that the series expansion of quantum field theory in the Feynman diagrams can be explicitly mapped on the partition function of the simplicial string theory – the theory describing embeddings of the 2D simplicial complexes into the space-time of the field theory. The summation over 2D geometries in this theory is obtained from the summation over the Feynman diagrams and the integration over the Schwinger parameters of the propagators. We discuss the meaning of the obtained relation and derive the 1D analog of the simplicial theory on the example of the free relativistic particle.

PACS: 11.25.–w

1. There is a hope that the large N Yang–Mills theory is exactly equivalent to a string theory [1]. Such a string theory if present can reveal the integrability of the large N Yang–Mills theory. Hence, the theory will help in explaining the confinement phenomenon.

Despite the recent progress [2, 3] for the case of (super)conformal theories we still do not understand the relation between gauge and string theories. We do not understand which features of the relation are generic (persist at least for the non-conformal and/or non–supersymmetric cases) and which are specific for the concrete relation of [2]. This is due to the fact that there is no explicit proof of the AdS/CFT correspondence [2, 3].

In an attempt to understand the relation between field and string theories in general we present, in this note, the explicit map from the functional integral of the matrix field theory (at finite N) onto the partition function of the simplicial string theory – the theory describing embeddings of the 2D simplicial complexes into the space-time of the field theory. Our considerations are quite generic and can be applied to the Yang–Mills theory. However, we consider the model example of the matrix Φ^3 theory whose interpretation on the string theory side we understand best of all.

The map in question is given by a duality transformation. To some extent this duality is the lattice analog of the T-duality map, although we do not have any compact dimensions. Via this transformation we map the summation over the Feynman diagrams and the integration over the Schwinger parameters onto the sum over the triangulations of the 2D surfaces and the integration over the invariant 2D distances between the vertices of the simplecial comlexes. This seems to be a summation over *all* 2D geometries and *all* embeddings of the simplicial complexes into the space–time. To understand this point we consider the toy example of the free relativistic particle, for which we present a similar expression. There, the summation over *all* 1D geometries is given by the summation over 1D "triangulations" and integrations over the lengths between the vertices of the "triangulations". The integration over all positions of the vertices gives the sum over *all* possible embeddings. The resulting "triangulated" expression is *exactly equivalent* to the relativistic particle path integral: No continuum limit should be taken.

However, the complete understanding of the simplicial string theory – at least its possible continuum formulation, or may be a continuum limit of it – is still lacking. In particular, it is possible that in the continuum formulation the theory describes strings in the curved AdS_5 space rather than in R^4 [4].

Anyway, as usual, the relation between two theories can be useful to both of them. In fact, the map in question at least can give an unambiguous way of formulating the simplicial string theory. Particularly, the measure of integration and the 2D gravity action unambiguously follow from the matrix field theory.

The structure of the paper is as follows. In the second section we present the map between the two theories. In the third section we present interpretation of the resulting dual expression obtained in the second section. In the fourth section we consider the example of the free relativistic particle and present a simplicial path integral for it. We conclude with the discussion in the fifth section. In the Appendix we present a simple proof of the well known combinatoric formulae [5] for the Feynman integrals. These formulae acquire a new meaning after

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the relation of the field theory to the simplicial string theory is established.

2. Consider the matrix scalar field theory in the D-dimensional Euclidian space:

$$Z = \int D\widehat{\Phi}(x) \times$$
$$\times \exp\left\{-\int d^D x N \mathrm{Tr}\left[\frac{1}{2}|\boldsymbol{\partial}\widehat{\Phi}|^2 + \frac{1}{2}m^2|\widehat{\Phi}|^2 + \frac{\lambda}{3}\widehat{\Phi}^3\right]\right\}, \quad (1)$$

where $\boldsymbol{\partial} = (\partial/\partial x_1, \dots, \partial/\partial x_D)$, $\widehat{\Phi}$ is $N \times N$ matrix field in the adjoint representation of $U(N)$ group: Φ^{ij}, $i, j = 1, \dots, N$. Note that we have re–scaled the fields so that λ is the 't Hooft coupling constant, but we are *not* taking the large N limit in this note.

The problems of this field theory, due to the sign indefiniteness of the Φ^3 potential, are not relevant for the most of our further considerations: We consider the functional integral Z as a formal series expansion in the powers of λ. To deal with connected graphs we consider $\log Z$.

It is well known that $\log Z$ can be represented as (see e.g. [6, 7]):

$$\log Z = \sum_{g=0}^{\infty} N^{\chi(g)} \sum_{V=0}^{\infty} \lambda^V C(V,g) \times$$
$$\times \int_0^{+\infty} \prod_{n=1}^{L} d\alpha_n \int \prod_{i=1}^{V} d^D \mathbf{y}_i \int \prod_{m=1}^{L} d^D \mathbf{p}_m \times$$
$$\times \exp\left\{-\sum_{l=1}^{L}\left[\frac{\alpha_l\,(\mathbf{p}_l^2+m^2)}{2} - \mathrm{i}\,\mathbf{p}_l\,(\Delta_l \mathbf{y})\right]\right\}, \quad (2)$$

where $\mathbf{p}_l$ is the momentum running along the propagator l; the propagators are written in the Schwinger α–representation; the first sum is taken over the genera g of the discretized closed 2D surfaces represented by the fat Feynman diagrams[2)] [7]; the second sum is taken over the number V of the insertions of $\mathrm{Tr}\widehat{\Phi}^3(\mathbf{y}_i)$ vertices; $\chi(g) = V - L + F$ is the Euler characteristic corresponding to the genus g diagram in the sum with V vertices, L propagators and F faces[3)]; $\Delta_l \mathbf{y}$ is the difference of the target space positions of the ends of the l–th propagator; $C(V,g)$ are constants which can be found from the genus expansion of the matrix integrals (see e.g. [8], [9]):

$$\int d\widehat{\Phi}\, \exp\left\{-N\,\mathrm{Tr}\left[\frac{1}{2}\,\widehat{\Phi}^2 + \frac{\lambda}{3}\widehat{\Phi}^3\right]\right\} =$$
$$= \sum_{g=0}^{\infty} N^{\chi(g)} \sum_{V=0}^{\infty} \lambda^V\, C(V,g). \quad (3)$$

For the general D most of the integrals under the sum in eq. (2) are divergent. One of the types of the divergences is proportional to the volume of the space-time and is just due to the translational invariance. To get rid of this divergence we can skip one of the L integrations over the momenta. Another type of the divergences are the standard Ultra Violet (UV) divergences of quantum field theory. We discuss them below.

We are going to perform a transformation over eq. (2). The same kind of transformation is performed in [10] and is referred to as duality on the lattice. As well somewhat similar transformation is made in [11] and relates some types of the Feynman diagrams of the Φ^3 theory to the amplitudes in conformal Quantum Mechanics.

To do this transformation let us perform the integration over the y's. Then each term under the sum and integration over α's is represented as the *finite* function:

$$I(L, V, \{\alpha\}) = \int \prod_{m=1}^{L} d^D \mathbf{p}_m \prod_{i=1}^{V} \delta\left(\sum_{l(\to i)}^{3} \mathbf{p}_l\right) \times$$
$$\times \exp\left\{-\sum_{l=1}^{L} \frac{\alpha_l\,(\mathbf{p}_l^2+m^2)}{2}\right\}, \quad (4)$$

where in each of the V δ–functions the sum goes over the three links terminating on each of the V vertices. These are momentum conservation conditions at each vertex. The UV divergences in the diagrams appear after the integrations over the α's. To perform the transformation in question we consider integrand expressions, because we would like to show that this transformation gives a non–trivial relation between the two partition functions rather than a formal map from one infinite number onto another. At the same time the divergences have a clear physical meaning on the both sides of the realtion as is argued in the next section.

The conditions which are imposed by the V δ-functions are usually solved via $G = L - V + 1$ independent momenta running along the loops of the diagram. However, we are going to solve them via the dual graph to the Feynman diagram under consideration. The dual graph consists of the vertices sitting in the centers of the faces of the Feynman diagram and its links are passing through the centers of the propagators of the Feynman diagram. Thus, dual graph to a three-valent Feynman diagram represents the triangulation of a 2D

[2)] Each member in the sum in eq. (2) is represented by the fat three–valent (three links are entering each of the V vertices) graph. Such a closed graph represents a vacuum amplitude of the theory in eq. (1). The generalization of our considerations for the correlators – open graphs – is straightforward.

[3)] Do not confuse this number with the number G of the momentum loops of the diagram; F is the number of the closed index loops of the *fat* Feynman diagram.

surface: The faces of the dual graphs are triangles.

Then each of the L momenta $\mathbf{p}_l$ obeying conditions $\sum_{l(\to i)}^{3} \mathbf{p}_l = 0$ can be represented as[4]:

$$\mathbf{p}_l = \Delta_l \mathbf{x} + \sum_{s=1}^{2g} \boldsymbol{\mu}_s \omega_l^{(s)}, \tag{5}$$

where $\Delta_l \mathbf{x}$ is the difference of the target space positions of the ends of the link l of the dual graph (which is intersecting the l–th propagator of the Feynman diagram); $\boldsymbol{\mu}_s$ are arbitrary parameters and $\omega_l^{(s)}$ are $2g$ closed (but not exact) one–forms on the genus g simplicial complex defined by the dual graph. To explain these observations let us point out that the condition $\sum_{l(\to i)}^{3} \mathbf{p}_l = 0$ is equivalent to the D 2D $d\mathbf{p} = 0$ conditions on the lattice [10]. The solutions of these conditions are $\mathbf{p} = d\mathbf{x} + \sum_{s=1}^{2g} \boldsymbol{\mu}_s \omega^{(s)}$, $d\omega^{(s)} = 0$ for all s whose lattice expression is eq. (5).

Using this solution we obtain:

$$\log Z = \sum_{g=0}^{\infty} N^{\chi(g)} \sum_{V=0}^{\infty} \lambda^V C'(V,g) \times$$
$$\times \int \prod_{s=1}^{2g} d^D \boldsymbol{\mu}_s \left[\det\left(\sum_{n,m=1}^{L} \omega_n^{(s)} \omega_m^{(s')}\right)\right]^{D/2} \times$$
$$\times \int_0^{+\infty} \prod_{n=1}^{L} d\alpha_n \int \prod_{a=1}^{F} d^D \mathbf{x}_a \exp\left\{-\sum_{l=1}^{L} \frac{\alpha_l}{2} \times\right.$$
$$\left.\times \left[(\Delta_l \mathbf{x})^2 + \left(\sum_{s=1}^{2g} \boldsymbol{\mu}_s \omega_l^{(s)}\right)^2 + m^2\right]\right\} =$$
$$= \sum_{g=0}^{\infty} N^{\chi(g)} \sum_{V=0}^{\infty} \lambda^V C'(V,g) \int_0^{+\infty} \prod_{n=1}^{L} \frac{d\alpha_n}{\alpha_n^{gD}} e^{-\frac{\alpha_n m^2}{2}} \times$$
$$\times \int \prod_{a=1}^{F} d^D \mathbf{x}_a \exp\left\{-\sum_{l=1}^{L} \frac{\alpha_l}{2} (\Delta_l \mathbf{x})^2\right\}, \tag{6}$$

where $C'(V,g)$ is different from $C(V,g)$ by the $D/2$ power of the determinant of the matrix relating p's and x's in eq. (5); F is the number of the vertices (faces) of the dual (Feynman) graph. In eq. (6) we have used the fact that ω's are closed.

In the next section we interpret the expression in eq. (6) as the simplicial string theory. In this context the summations over the genera, triangulations and the integrations over the α's give the summation over internal 2D geometries. The integration over the x's – positions of the vertices – gives the summation over the embeddings.

It is worth mentioning at this point that all our considerations so far can be easily generalized to higher valent fat graphs (i.e. to the matrix Φ^n, $n \geq 4$ theory or to non-Abelian gauge theories). However, the resulting dual graphs in these cases contain more complicated simplexes than just triangles [12].

3. The definition of the simplicial string theory is well known [13]. We present it here to make the interpretation of eq. (6) obvious. First, the internal metric on a simplicial complex is given by:

$$||h_{\alpha\beta}||_\nabla = ||\mathbf{e}_\alpha \mathbf{e}_\beta||_\nabla = \begin{pmatrix} \mathbf{e}_1^2 & \mathbf{e}_1 \mathbf{e}_2 \\ \mathbf{e}_1 \mathbf{e}_2 & \mathbf{e}_2^2 \end{pmatrix} =$$
$$= \begin{pmatrix} e_1^2 & \frac{1}{2}\left[e_1^2 + e_2^2 - e_3^2\right] \\ \frac{1}{2}\left[e_1^2 + e_2^2 - e_3^2\right] & e_2^2 \end{pmatrix}, \tag{7}$$

where $\mathbf{e}_\alpha$, $\alpha = 1,2$ are 2D vectors which are setting the zweibein. They are along two edges of each triangle ∇ of the simplicial complex. As well $e_{1,2,3}$ are lengths of the three edges of these triangles. Second, the external metric on a simplicial complex is given by:

$$||G_{\alpha\beta}||_\nabla = \begin{pmatrix} (\Delta_1 \mathbf{x})^2 & \Delta_1 \mathbf{x}\, \Delta_2 \mathbf{x} \\ \Delta_1 \mathbf{x}\, \Delta_2 \mathbf{x} & (\Delta_2 \mathbf{x})^2 \end{pmatrix} = \begin{pmatrix} (\Delta_1 \mathbf{x})^2 & \frac{1}{2}\left[(\Delta_1 \mathbf{x})^2 + (\Delta_2 \mathbf{x})^2 - (\Delta_3 \mathbf{x})^2\right] \\ \frac{1}{2}\left[(\Delta_1 \mathbf{x})^2 + (\Delta_2 \mathbf{x})^2 - (\Delta_3 \mathbf{x})^2\right] & (\Delta_2 \mathbf{x})^2 \end{pmatrix}. \tag{8}$$

$\Delta_{1,2,3}\mathbf{x}$ are differences of the target space positions of the vertices of each triangle of the simplicial complex. Hence, the discretization of the string theory action is as follows:

[4] Note that the momenta under the sums $\sum_{l(\to i)}^{3} \mathbf{p}_l = 0$ have alternating signs: Some momenta are entering the vertex, while the others are exiting from it. This obviously means that the links of the Feynman diagram have orientations. Hence, the links of the dual graph should have synchronized orientations (with the Feynman diagram) which will define in eq. (5) which of the edge x's enters with "+" and which with the "-" sign in the corresponding p so that all the momentum conservation conditions are fulfilled [12].

$$S = \int d^2\sigma \sqrt{h}\, h^{ab}\, \partial_a \mathbf{x}\, \partial_b \mathbf{x} => \sum_{\nabla} \sqrt{h_\nabla}\, \mathrm{Tr}\left(||h||_\nabla^{-1}\, ||G||_\nabla\right) =$$
$$= \sum_{\nabla} \frac{\left[(\Delta_1 \mathbf{x})^2 \left(e_2^2 + e_3^2 - e_1^2\right) + (\Delta_2 \mathbf{x})^2 \left(e_1^2 + e_3^2 - e_2^2\right) + (\Delta_3 \mathbf{x})^2 \left(e_1^2 + e_2^2 - e_3^2\right)\right]_\nabla}{2\sqrt{h_\nabla}}. \quad (9)$$

Here the sum is going over all triangles of a simplicial complex and

$$h_\nabla = \frac{1}{4}\left[\left(e_2^2 + e_3^2 - e_1^2\right) \left(e_1^2 + e_3^2 - e_2^2\right) + \left(e_2^2 + e_3^2 - e_1^2\right) \left(e_1^2 + e_2^2 - e_3^2\right) + \left(e_1^2 + e_2^2 - e_3^2\right) \left(e_1^2 + e_3^2 - e_2^2\right)\right]_\nabla \quad (10)$$

is the determinant of the internal metric. Thus, it is natural to define the partition function of the simplicial string theory as:

$$Z_{sst} = \sum_{\mathrm{Triangulations}} \int [de]_{\mathrm{Triangulation}}\, e^{-S(e)} \prod_{a=1}^{F} d^D \mathbf{x}_a\, \exp\left\{-\sum_{l=1}^{L} \frac{\alpha_l(e)\, (\Delta_l \mathbf{x})^2}{2}\right\}, \quad (11)$$

where F is the number of vertices of the triangulation under the sum; $\alpha_l(e)$, as follows from eq. (9) and eq. (10), are the positive functions of the lengths of the edges of the two triangles glued together via the link l. What is left to be defined is the measure $[de]$ and the weight $S(e)$ of the summation over the 2D geometries. If we would like to integrate over the e's themselves we have to impose the triangle inequalities into the measure to keep the metric positive defined.

Now we can point out the equivalence between eq. (6) and eq.(9)–eq.(11) with the suitable choice of $[de]$ and $S(e)$. In fact, the measure and the weight for the summation over the 2D geometries in eq. (6) unambiguously follows from the matrix field theory. This measure is very natural because the integration goes over the α's rather than e's which demand triangle inequalities to be imposed [12]. However, the expressions for the discretized versions of the standard gravity actions in terms of the α's are not known. This explains the reason why usually in the formulation of the simplicial string theory one is trying to express everything through the e's rather than the α's [13].

It is worth mentioning at this point that the UV divergences of the quantum field theory in eq. (1) acquire a clear interpretation in the simplicial string theory description. These divergences are just due to the boundaries in the space of all metrics: I.e. due to the degenerate metrics, which correspond to such situations when some of the triangles degenerate into links. In this context it is interesting to understand the meaning of the renormalization group within the simplicial string theory context (see [14] for the attempts of the explanation).

Note that eq. (11) and eq. (6) are explicitly reparametrization invariant, because there the integration is going over all reparametrization invariant 2D lengths between the vertices of the simplicial complexes and over the target space positions x's of the vertices rather than over the maps [12]. In the next section we will present similar situation for the relativistic particle. After that we will be ready for the discussion of the 2D situation.

4. Consider the path integral for the relativistic particle:

$$G(\mathbf{x}, \mathbf{x}') = \int_{\mathbf{x}}^{\mathbf{x}'} D\mathbf{x}(\tau) \int \frac{De(\tau)}{VolDiff} \times$$
$$\times \exp\left\{-\frac{1}{2}\int_0^1 d\tau \left[\frac{\dot{\mathbf{x}}^2}{e(\tau)} + m^2\, e(\tau)\right]\right\} \quad (12)$$

with the measures following from the norms:

$$||\delta \mathbf{x}(\tau)||^2 = \int_0^1 d\tau\, e(\tau)\, [\delta \mathbf{x}(\tau)]^2 = T \int_0^1 df [\delta \mathbf{x}(f)]^2,$$
$$||\delta e(\tau)||^2 = \int_0^1 d\tau\, e(\tau) \left[\frac{\delta e(\tau)}{e(\tau)}\right]^2. \quad (13)$$

The answer for this path integral is [1]:

$$G(\mathbf{x}, \mathbf{x}') \propto \int_0^\infty \frac{dT}{\sqrt{T}}\, \mathrm{det}^{\frac{1-D}{2}} \left(-\frac{1}{T^2}\frac{d^2}{df^2}\right) \times$$
$$\times \exp\left\{-\frac{1}{2}\left[\frac{(\mathbf{x}-\mathbf{x}')^2}{T} + m^2\, T\right]\right\}. \quad (14)$$

In the ζ–function regularization we obtain:

$$G(\mathbf{x},\mathbf{x}') = \\ = \int\limits_0^{+\infty} \frac{dT}{T^{D/2}} \exp\left\{-\frac{1}{2}\left[\frac{(\mathbf{x}-\mathbf{x}')^2}{T} + m^2\,T\right]\right\} = \\ = \int \frac{d^D\mathbf{p}}{(2\pi)^D}\frac{e^{i\mathbf{p}\,(\mathbf{x}-\mathbf{x}')}}{\mathbf{p}^2+m^2}. \quad (15)$$

Thus, $G(\mathbf{x},\,\mathbf{x}')$ is the Green's function of the Klein–Gordon equation.

At the same time there is another reparametrization invariant regularization for the path integral of the relativistic particle: The lattice regularization, where the lattice spacings are reparametrization invariant one–lengths. In this regularization naively one has ($T = \sum_{i=0}^{M} e_i$):

$$\det^{-\frac{1}{2}}\left(-\frac{1}{T^2}\frac{d^2}{df^2}\right) = \\ = \int D\lambda(\tau)\,\exp\left\{-\int\limits_0^1 \frac{\dot\lambda^2(\tau)}{2\,e(\tau)}\,d\tau\right\} \Rightarrow \\ \Rightarrow \int \sqrt{e_0}\prod_{i=1}^{M}\sqrt{e_i}\,d\lambda_i\,\exp\left\{-\sum_{j=0}^{M}\frac{(\lambda_{j+1}-\lambda_j)^2}{2e_j}\right\} \propto \\ \propto \frac{\prod_{i=0}^{M} e_i}{T^{\frac{1}{2}}}, \quad (16)$$

where

$$||\delta\lambda||^2 = \int\limits_0^1 d\tau e(\tau)\;[\delta\lambda(\tau)]^2 \Rightarrow \sum_{0=1}^{M} e_i\;(\delta\lambda_i)^2\,, \\ e_i = \int\limits_{i\,\Delta\tau}^{(i+1)\,\Delta\tau} e(\tau)\,d\tau,$$

and $\delta\lambda_0 = \delta\lambda_{M+1} = 0$. Note that e's are invariant 1D lengths. If this expression for the determinant is substituted into eq.(14) we obtain:

$$G_{Latt}(\mathbf{x},\,\mathbf{x}') = C_M \int \frac{d^D\mathbf{p}}{(2\,\pi)^D}\frac{e^{i\,\mathbf{p}\,(\mathbf{x}-\mathbf{x}')}}{(\mathbf{p}^2+m^2)^{M-DM/2}}. \quad (17)$$

Here C_M is some constant.

It seems that the naive lattice regularization does not work. However, for this case we can present a multiple integral expression which solves the Klein–Gordon equation [12]. To obtain it note that in contrast with respect to the evolution type equations[5)] the Green's function of the Klein–Gordon equation has the following feature:

$$\int \prod_{i=1}^{M} d^D\mathbf{y}_i\,G(\mathbf{x},\,\mathbf{y}_1)\,G(\mathbf{y}_1,\,\mathbf{y}_2)...\,G(\mathbf{y}_M,\,\mathbf{x}') = \\ = \int \frac{d^D\mathbf{p}}{(2\pi)^D}\frac{e^{i\,\mathbf{p}\,(\mathbf{x}-\mathbf{x}')}}{(\mathbf{p}^2+m^2)^{M+1}} \neq G(\mathbf{x},\,\mathbf{x}'). \quad (18)$$

However, it is easy to correct this formula in such a way that the equality will hold. For example, we can put (for any M):

$$G(\mathbf{x},\,\mathbf{x}') \propto \int \prod_{i=1}^{M} d^D\mathbf{y}_i \prod_{j=0}^{M}\frac{d^D\mathbf{p}_j}{(2\pi)^D}\,d\left(\sum_{k=0}^{M} e_k\right)\times \\ \times \exp\left\{\sum_{m=0}^{M}\left[i\,\mathbf{p}_m\;(\mathbf{y}_{m+1}-\mathbf{y}_m)\;- \right.\right. \\ \left.\left. -\frac{1}{2}\;(\mathbf{p}_m^2+m^2)\;e_m\right]\right\} \propto \\ \propto \int \prod_{i=1}^{M} d^D\mathbf{y}_i\,\frac{d\left(\sum_{j=0}^{M} e_j\right)}{\prod_{n=0}^{M} e_n^{D/2}}\times \\ \times \exp\left\{-\frac{1}{2}\sum_{m=0}^{M}\left[\frac{(\mathbf{y}_{m+1}-\mathbf{y}_m)^2}{e_m} + m^2\,e_m\right]\right\}, \quad (19)$$

where $\mathbf{y}_0 = \mathbf{x}$, $\mathbf{y}_{M+1} = \mathbf{x}'$. In this formula we take the integral over the moduli $\sum_{i=0}^{M} e_i = T$ rather than over all e's and the expression under this integral depends on T rather than all e's separately. The latter fact can be seen explicitly after the integration over y's.

The expression in eq. (19) seems to be a good candidate for the "proper discretization" of the relativistic particle path integral. However, due to the integration over T rather than each separate e's this integral does not seem to have a good local field theory interpretation. Note that obviously:

$$\int d\left(\sum_{i=0}^{M} e_i\right)\cdots \neq \int \frac{\prod_{i=0}^{M} de_i}{VolDiff}\cdots \quad (20)$$

for any M. This is the main difference from the above case of the relativistic particle *path integral* when the limit $M = \infty$ is appropriately taken and the ζ–function regularization instead of the lattice one is applied.

To obtain the integration over all e's let us perform the following trick. Consider the equality:

[5)] Which obey:

$$K(\mathbf{x},\,\mathbf{x}'|(M+1)\,\Delta\tau) = \\ \int \prod_{i=1}^{M} d^D\mathbf{y}_i\,K(\mathbf{x},\,\mathbf{y}_1|\Delta\tau)\,K(\mathbf{y}_1,\,\mathbf{y}_2|\Delta\tau)...\,K(\mathbf{y}_M,\,\mathbf{x}'|\Delta\tau).$$

$$\frac{1}{\mathbf{p}^2+m^2} \propto \sum_{L=0}^{\infty} \frac{(-1)^L}{L!} \int\limits_0^{+\infty} \prod_{l=1}^{L} \frac{de_l}{e_l} \times \\ \times \exp\left\{-\frac{(\mathbf{p}^2+m^2)}{2} \sum_{n=0}^{L} e_n\right\}. \qquad (21)$$

Then it is possible to write (in each term under the sum $\mathbf{y}_0 = \mathbf{x}$, $\mathbf{y}_{L+1} = \mathbf{x}'$):

$$G(\mathbf{x},\mathbf{x}') = \sum_{L=0}^{\infty} \frac{(-1)^L C_L}{L!} \int\limits_0^{+\infty} \prod_{n=1}^{L} \frac{de_n}{e_n^{D/2+1}} \times \\ \times \int \prod_{i=1}^{L} d^D\mathbf{y}_i \exp\left\{-\frac{1}{2}\sum_{l=0}^{L}\left[\frac{(\Delta_l \mathbf{y})^2}{e_l} + m^2 e_l\right]\right\}, \quad (22)$$

where C_L are constants. The enumeration of the links in this 1D case coincides with the enumeration of the vertices: $\Delta_l \mathbf{y} = \mathbf{y}_{l+1} - \mathbf{y}_l$, $l = i$ and in the 1D (open path) case $L+1 = V$. If we take the integrals over y's instead of p's then the conditions are $\Delta_i \mathbf{p} = \mathbf{p}_{i+1} - \mathbf{p}_i = 0$ (for all i) whose continuum limit analogs are $d\mathbf{p} = \partial_\tau \mathbf{p}\, d\tau = 0$. The solution of the latter on the 1D interval is $\mathbf{p}(t) =$ const. That is the reason why, unlike the 2D case in eq. (6), in the 1D situation we do not have a non–trivial expression for the Green's function through the $\mathbf{x}_a$'s.

However, the formula in eq. (22) is in many respects very similar to the 2D expression in eq. (6). In fact, it contains the summation over all discretizations of the world–trajectory (which are 1D triangulations) and the integration over all 1D distances between the vertices y_i's (which is the integration over the α's). In the 1D case $\alpha(e) = e$. The summation over the embeddings is given by the integration over all possible positions of the vertices (y's).

5. Thus, we find that the log of the functional integral for the matrix quantum field theory can be represented as the partition function of the first quantized simplicial string theory. In the latter we sum over all possible embeddings of all possible simplicial complexes into the target space. Instead of the summation over the 2D metrics we sum over all possible triangulations and invariant 2D distances between the vertices of the simplicial complexes. Both of them seem to be summations over all 2D geometries. At the same time the action describing embeddings of the simplicial complexes appears to be the discretization of the standard Polyakov action for the relativistic string theory in the flat space [1].

In an attempt to understand the resulting simplicial string theory we consider the relativistic particle case. Here we have two *equivalent* expressions: eq. (12) and eq. (22). One of them includes integration over all smooth 1D metrics, while the other expression contains, in effect, integration over all singular 1D metrics. Both of them are containing summations over all 1D geometries with the fixed topology (open paths). Note that it is not necessary to take a continuum limit in eq. (22) to obtain the correct solution to the Klein–Gordon equation. It is not even clear how to take a continuum limit in the expression like eq. (22). In fact, taking $L = \infty$ does not mean the continuum limit.

Similarly to the 1D case the 2D expression in eq. (6) is explicitly reparametrization invariant and seems to include the summation over all 2D geometries. Then it is tempting to find exactly equivalent to it a continuum expression containing integration over all smooth metrics. This looks like a crazy idea. At least there is no good reason why $\lambda^V\, C(g,V)$ are the appropriate constants for the equality to be true with a suitable measure for the smooth metrics: There is no freedom for the choice of C_L's in eq. (22).

Frankly speaking, we do not know whether the aforementioned temptation is meaningful or that it is necessary to take a continuum limit, whatever it means. Possible reason for taking a continuum limit can be as follows [12]. In the 1D case we have a singled out point as the boundary of the world trajectory rather than a curve – continuous sequence of points. Hence, the equation for the path integral following from the variation of the boundary point is just a differential equation. At the same time the generalization of this differential equation to the 2D case is loop equation on the boundary curve. Apart from that there seems to be another risk: The gravity action after the change from α's to e's can appear to be non-local. However, we do not think that this is the case. In fact, the non-locality if present should be rather trivial because the change from α's to e's is local (depends on adjacent triangles) and the measure in eq. (6) depends on α's locally (it is the product over the triangles). In any case this question demands a separate investigation.

Anyway, we believe that the choice among the two possibilities can be made after the derivation/understanding the meaning of the loop equation or its discretized version for the open string theory in eq. (6).

I would like to acknowledge valuable discussions with A. Rosly, A. Morozov, N. Amburg, A. Isaev, A. Gorsky, T. Pilling, Yu. Makeenko and D. Malyshev. Especially I would like to thank V. Dolotin for intensive collaboration and sharing his ideas. This work was done under the partial support of grants RFBR # 02-02-17260, INTAS # 03-51-5460 and the Grant from the President of Russian Federation MK-2097.2004.2.

Appendix: Calculations of the graphs. In this Appendix we sketch a proof of some combinatorial formulae for the Feynman diagrams which have a clear meaning within the context of the simplicial string theory. These formulae were proved in [5] using the electric net analogy [15]. Our proof is purely combinatoric and less tedious.

Consider a Feynman diagram in any scalar quantum field theory (with standard – polynomial in fields – interactions) which has V_E external vertices, V_I internal vertices and L propagators. The positions of the external vertices are z_a, $a = 1, ..., V_E$. All propagators are written in the Shwinger α–representation. The expression for this diagram $I(\mathbf{z}_1, ..., \mathbf{z}_{V_E})$ gives a quantum field theory amplitude. We would like to represent the integrand expression under the integration over the α's explicitly in a combinatoric form [5].

To calculate this diagram let us present the recurrent relation between the graphs [12]. Consider a complete graph (all vertices of which are connected by links to each other) with V vertices. Assign to this graph the following expression:

$$F_V\left(\{\mathbf{z}_a\}\middle|\{\alpha_{(ab)}\}\right) = \prod_{a\neq b}^{V} \frac{1}{\alpha_{(ab)}^{D/2}} \exp\left\{-\frac{\left(\Delta_{(ab)}\mathbf{z}\right)^2}{2\alpha_{(ab)}}\right\}. \quad (23)$$

Let us take the integral say over $\mathbf{z}_V$. The result is ($\beta = 1/\alpha$):

$$\int d^D\mathbf{z}_V\, F_V\left(\{\mathbf{z}_a\}\middle| \quad \{\beta^{-1}_{(ab)}\}\right) =$$

$$= \frac{\left(\prod_{a\neq b}^{V}\beta_{(ab)}\right)^{D/2}}{\left(\prod_{a'\neq b'}^{V-1}\tilde{\beta}_{(a'b')}\right)^{D/2}} \frac{1}{\left(\sum_{c'=1}^{V-1}\beta_{(c'V)}\right)^{D/2}} \times$$

$$\times F_{V-1}\left(\{\mathbf{z}_{a'}\}\middle| \quad \{\tilde{\beta}^{-1}_{(a'b')}\}\right), \quad (24)$$

where:

$$\tilde{\beta}_{(a'b')} = \beta_{(a'b')} + \frac{\beta_{(a'V)}\,\beta_{(Vb')}}{\sum_{c'=1}^{V-1}\beta_{(c'V)}}, \qquad a', b' = 1, ..., V-1. \quad (25)$$

Along this way we can obtain expression for any graph. In fact, choosing big enough V, taking $\beta \to 0$ (or $\alpha \to \infty$) for the missing links in the graph and making the appropriate number of the integrations over z's, we can always do that. The resulting expression has a clear combinatoric representation [5]. This expression is easy to see by induction from the presented here formulae. In particular, the resulting expression for the aforementioned Feynman diagram is [5]:

$$I\left(\mathbf{z}_1, ...\mathbf{z}_{V_E}\right) \propto \int_0^{+\infty} \prod_{l=1}^{L} d\beta_l\, \beta_l^{D/2-2} \frac{1}{\Delta(\beta)^{D/2}} \times$$

$$\times \exp\left\{-\sum_{n=1}^{L}\frac{m^2}{2\,\beta_n} - \frac{P(\beta, \mathbf{z})}{4}\right\}. \quad (26)$$

Here:

$$\Delta(\beta) = \sum_{t_1}\prod^{V_I}\beta,$$

$$P(\beta, \mathbf{z}) = \frac{\sum_{t_2}\left(\prod^{V_I+1}\beta\right)\left(\Delta_{t_2}\mathbf{z}\right)^2}{\Delta(\beta)}, \quad (27)$$

where in the first expression the sum is going over all so called dual-trees t_1 of the diagram, while in the second expression the sum is going over all dual-2-trees t_2 of the diagram. In these expressions we take products of β's along the corresponding dual-co-trees and dual-co-2-trees correspondingly; $\Delta_{t_2}\mathbf{z}$ is the difference of the positions of the two external vertices that come together in a dual–2–tree t_2.

The definition of all these "dual-(co)-(2)-trees" is as follows [5]. The tree graph (not necessary connected) obtained by shrinking V_I lines of the diagram such that all V_I internal vertices merge with the external vertices, but that no pair of external vertices become coincident, is called a dual-tree; the set of V_I shrunk lines a dual-co-tree. If we shrink $V_I + 1$ lines so that not only all the internal vertices merge with the external ones, but also exactly two external vertices come together, then the resulting graph is a dual-2-tree; the set of $V_I + 1$ shrunk lines a dual-co-2-tree.

The reason why we present these formulae here is the following. The same kind of formulae can be written for the dual graph to a Feynman diagram. For the dual graph such an amplitude has the meaning of ether a scattering amplitude of closed strings or an open string amplitude. Hence, for this dual graph $\Delta(\beta)$ and $P(\beta, \mathbf{z})$ are related to the determinant of the discretized 2D Laplacian (in curved metric) and 2D classical action (with the boundary conditions given by z's) correspondingly [12]. But this is a theme for a separate scientific investigation.

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