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State-dependent dynamical variables in quantum theory

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State-dependent local dynamical variables (LDV) sharply differ from the ordinary operators of quantum mechanics. The N-level model system shows the physical importance of such operators in the complex projective Hilbert state space $CP(N-1)$. The process of the quantum measurement in terms of the LDV is described.

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In our macroscopic experience we have the solid pseudo-Euclidian space-time structure. Physical conservation laws rid us from doubts about the macroscopic system identification. The identification of the quantum system at very short and at cosmic distances is not so simple. How may one be sure that, say, a right helicity photon has been send by Alice? Physically this question may be formulated as follows: what is an objective criterion for the identity of quantum system, or, what is the physical mechanism of the self-identification (self-conservation) of a quantum system? We can no more rely upon the space-time symmetries since just these properties should be established in some approximation a posteriori. In such a situation one should have some conservation laws relying upon the geometry of the intrinsic transformation groups and its sub-manifolds.

I formulate the covariant dynamics of the N-level quantum system and corresponding local dynamical variables (LDV) based purely on the $SU(N)$ geometry. Since the quantum states are rays, in fact only the transformations from the coset sub-manifold $G/H = U(N)/U(1) \times U(N-1) = CP(N-1)$ act effectively on the rays of states. However LDV (defined in terms of tangent vectors to $CP(N-1)$) being expressed in terms of local quantum coordinates $(\pi^1 = \Psi^1/\Psi^0, \dots, \pi^{N-1} = \Psi^{N-1}/\Psi^0)$ are subject to the action of whole $SU(N)$ group. Hence we may assume that $SU(N)$ transformations of the N-level quantum system are locally equivalent to a definite motion of the LDV in $CP(N-1)$ (the "super-relativity" principle [1]). Therefore parallel transport in $CP(N-1)$ is the method of N-level quantum system identification, expressing the conservation law of the LDV, should be observable and curvature-dependent.

1. I would like attract attention to the global property of the internal symmetries in quantum mechanics. They have been realized mostly in imitative of the form of the space-time symmetries. Namely, their operators give a linear representation of the corresponding groups. To my mind the nonlinear realization seems to be actually capable of shedding light on the measurement as an objective process [2].

In fact the nonlinear group realizations have been already used in the framework of the phenomenological Lagrangians method in QFT [3] and in the theory of spin wave interaction [4]. The breakdown of the "chiral" dynamical group $SU(N)$ up to the isotropy group $H = U(1) \times U(N-1)$ was proposed [1] but in an abstract form without clear physical argumentation. I will now show that in simple optical measurement the state-dependent LDV play the key role in the objective interpretation of the quantum theory.

My aim is to calculate the phase difference accumulated during the parallel transport of the LDV corresponding to light polarization along different paths in $CP(1)$. Let me describe the polarization optics measurement in the terms of LDV. The model setup providing the unitary evolution of the polarization state of light is simple. A fixed Cartesian reference frame (O, x, y, z) in physical space will be used. Initially one has a beam of light in a linear polarization state in x-direction $|x\rangle = \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle) = \frac{1}{\sqrt{2}}(1,1)^T$ propagating along z-axes. Then the polarization states in the y-direction is $|y\rangle = \frac{-i}{\sqrt{2}}(|R\rangle - |L\rangle) = \frac{1}{\sqrt{2}}(-i,i)^T$, and then $|R\rangle = (1,0)^T, |L\rangle = (0,1)^T$. The coherent superposition state will be denoted as usual $|\Psi\rangle = (\Psi^0, \Psi^1)^T$. The Poincaré sphere refers to the coordinates (o, s_1, s_2, s_3) in the iso-space of the polarization. In general the coherence vector lies on the isotropy "light cone" $s_0^2 - s_1^2 - s_2^2 - s_3^2 = 0$ where $s_0^2 = I^2 = \langle\Psi|\Psi\rangle$ is the square of the beam intensity. It means the coherence vector may fall into the Poincaré sphere under non-unitary evolution. I will restrict myself to the unitary one.

The initial state $|x\rangle$ is modulated passing through an optically active medium (say using the Faraday effect in YIG film magnetized along the main axes in the z-direction by a harmonic magnetic field with frequency Ω and the angle amplitude β). Formally this process may be described by the action of the unitary matrix $\hat{h}_{os_3}$ belonging to the isotropy group of $|R\rangle$ [1]. Then the coherence vector will oscillate along the equator of the Poincaré sphere. The next step is the dragging of the oscillating state $|x'(t)\rangle = \hat{h}_{os_3}|x\rangle$ with frequency ω up to the "north pole" corresponding to the state $|R\rangle$. In fact this is the motion of the coherence vector. This may be achieved by the variation of the azimuth of the linear polarized state from $\theta/2 = -\pi/4$ up to $\theta/2 = \pi/4$ with help of the dense flint of appropriate length embedded into the sweeping magnetic field. Further this beam should pass the $\lambda/4$ plate. This process of variation of the ellipticity of the polarization ellipse may be described by the unitary matrix $\hat{b}_{os'_1}$ belonging to the coset homogeneous sub-manifold $U(2)/U(1) \times U(1) = CP(1)$ of the dynamical group $U(2)$ [1]. This dragging without modulation leads to the evolution of the initial state along the geodesic of $CP(1)$ and the trace of the coherent vector is the meridian of the Poincaré sphere between the equator and one of the poles. The modulation deforms both the geodesic and the corresponding trace of the coherence vector on the Poincaré sphere during such unitary evolution.

The action of the $\lambda/4$ plate depends upon the state of the incoming beam state (the relative orientation of the fast axes of the plate and the polarization of the beam). Furthermore, only relative phases and amplitudes of photons in the beam have a physical meaning for the $\lambda/4$ plate. Neither the absolute amplitude (intensity of the beam), nor the general phase affect the polarization character of the outgoing state. It means that the device action depends only upon the local coordinates $\pi^1 = \Psi^1/\Psi^0 \in CP(1)$. Small relative re-orientation of the $\lambda/4$ plate leads to a small variation of the outgoing state. This means that the $\lambda/4$ plate re-orientation generates the tangent vector to $CP(1)$. It is natural to discuss the two components of such a vector: velocities of the variations of the ellipticity and of the azimuth (inclination) angle of the polarization ellipse. They are examples of LDV. The comparison of such dynamical variables for different coherent states requires that affine parallel transport agrees with the Fubini-Study metric. The deep reason for this is as follows.

2. Let us assume that initial state is $(\pi^1_A, \dots, \pi^{N-1}_A)$ and the final state is $(\pi^1_B, \dots, \pi^{N-1}_B)$. The state $(\pi^1_B, \dots, \pi^{N-1}_B)$ may be reached from any different state $(\pi^1_{A'}, \dots, \pi^{N-1}_{A'})$. In order to know the source of this state there should be some mark or "key". This may be achieved by establishing a set of constrains which an observer agrees to judge as "full enough" for the identification. Here we have a subjective factor. But it may be avoided if we chose the *intrinsic invariants* of the $CP(N-1)$ geometry. Then the subjective element will disappear and, hence, one will have an objective criterion for the identification. Formally it is based upon Cartan's method of moving frame eliminates the necessity of the "second particle" as a reference frame for the "first" one [5]. Generally the "minimally full" description of the quantum state in $CP(N-1)$ requires the adjoint representation of $SU(N)$ in R^{N^2-1} field parameter space. In fact these effective multipole fields describe the intensity of the device action. The tangent vector fields (differential operators) $D_\alpha = \Phi^i_\alpha \frac{\partial}{\partial \pi^i} + \text{c.c.}$ where Φ^i_α are as follows:

$$\Phi^i_\sigma = \lim_{\varepsilon \to 0} \varepsilon^{-1} \left\{ \frac{[\exp(i\varepsilon\lambda_\sigma)]^i_m \Psi^m}{[\exp(i\varepsilon\lambda_\sigma)]^j_m \Psi^m} - \frac{\Psi^i}{\Psi^j} \right\} = \\ = \lim_{\varepsilon \to 0} \varepsilon^{-1} \{\pi^i(\varepsilon\lambda_\sigma) - \pi^i\}. \quad (1)$$

These vector fields replace in my approach the Pauli matrices of $AlgSU(2)$, the Gell-Mann matrices of $AlgSU(3)$, etc. [1]. The LDV are state-dependent, i.e. local in $CP(N-1)$ and their expectation values (the scalar product in the sense of the Fubini-Study metric) are not bi-linear in general. Such expectation values are similar to an expectation value in the modified quantum mechanics of Weinberg [6]. The path-dependent parallel transport of the LDV in the affine connection

$$\Gamma^i_{mn} = \frac{1}{2} G^{ip^*} \left(\frac{\partial G_{mp^*}}{\partial \pi^n} + \frac{\partial G_{p^*n}}{\partial \pi^m} \right) = \\ = -\kappa \frac{\delta^i_m \pi^{n^*} + \delta^i_n \pi^{m^*}}{1 + \kappa \sum |\pi^s|^2} \quad (2)$$

agrees with the Kählerian metric (Fubini-Study metric)

$$G_{ik^*} = \frac{(1 + \kappa \sum |\pi^s|^2)\delta_{ik} - \kappa \pi^{i^*} \pi^k}{(1 + \kappa \sum |\pi^s|^2)^2} \quad (3)$$

as will be shown case of $CP(1)$. Here $\kappa = r^{-2}$ is the curvature of the sphere serving as a model of $CP(N-1)$ through the stereographic projection. I will assume temporary that $r = 1$ for simplicity.

The essential differences between my approach and, say, the approach of Anandan and Pati [7] are firstly, that I use the parallel transport of dynamical variables local in $CP(1)$ instead of the quantum state parallel transport. Secondly, the geometric frequency I use is local and it is applicable to any superposition state, whereas the Anandan-Pati "reference-section" of the state is bi-local and it is singular for the orthogonal initial and final states. Note, Berry's [8, 9] and the Aharonov-Anandan [10, 11] "parallel transport" laws of the quantum state are defined in original Hilbert space. This kind of the parallel transport is not an object of the intrinsic geometry of a parameter space (Berry) or the projective Hilbert state spaces (Aharonov-Anandan); see for example the explanation in [12]. Such a definition discards the dynamical phase shift and extracts the pure topological consequences of the rotations of polarizers, $\lambda/4$ plates, etc. However there are some reasons to keep dynamics together with geometry [10, 1, 13]. In particular the fundamental importance of the complex projective geometry of the state space $CP(N-1)$ [11, 14, 2, 1, 15, 13] strongly suggest working in the intrinsic geometry of $CP(N-1)$ associated with the quantum dynamics.

3. Now I introduce the parallel transporting real dynamical variable $T = T^1\frac{\partial}{\partial\pi^1} + T^{1*}\frac{\partial}{\partial\pi^{1*}}, T^{1*} = (T^1)^*$ assuming that T^1 obeys the following equations

$$\frac{dT^1}{dt} + \Gamma^1_{11}T^1\frac{d\pi^1}{dt} = 0, \quad \text{c.c.} \tag{4}$$

These equations have exact solutions along a geodesic of $CP(1)$: $T^1(s) = \xi(1+\tan^2(\omega t)) + i\eta(1+\tan^2(\omega t))$. The scalar product $G_{ik^*}T^i(s)T^{k*}(s) = \xi^2+\eta^2$ is the invariant of the parallel transport.

The modulation of the polarization plane orientation deforms the geodesic $\gamma(t)$ to $f(t)$. The equations (4) have for such path of the parallel transport only the numerical solutions which we shall call $\Xi^1(t)$, c.c.. Let me to show the difference between the parallel transported vectors $T^1(s)$ along the geodesic $\gamma(t)$ and the vector $S^1 = \Xi^1 - \Gamma^1_{11}\Xi^1 d\pi^1$ pointwise "shifted" from the deformed path $f(t)$ to the "reference" geodesic $\gamma(t)$ where $d\pi^1 = \pi^1(f(t)) - \pi^1(\gamma(t))$. It means all local tensors and Γ^p_{in} were calculated on the "reference" geodesic. The angle between these two vectors along the "reference" geodesic will be expressed through $\cos\chi(t)$:

$$\cos\chi(t) = \frac{|G_{ik^*}T^i(\gamma(t))S^{k*}(f(t))|}{\|T\|\|S\|}. \tag{5}$$

The cosine of the angle between the exact solution of the equation $T^i(\gamma(t))$ and the numerical solution $\Xi^{k*}(f(t))$ for the parallel transport along the deformed geodesic is shown in the picture Fig.1.

The result is very interesting: *all vectors parallel transported along different paths look like a smoothly opening "umbrella" along the geodesic. At $\theta = \pi/2$ the parallel transported dynamical variable along one of the deformed geodesics $f(t)$ are orthogonal (in the sense of the Fubini-Study metric) to the "handle" of the "umbrella"; the parallel transported vector along the geodesic. In fact this means that the result of the parallel transport is local: this is uniquely defined by the geodesic issuing from the initial point and by the dynamical variable (tangent vector). It looks, as a preliminary conclusion, like a "decoherence process" in the projective Hilbert state space.*

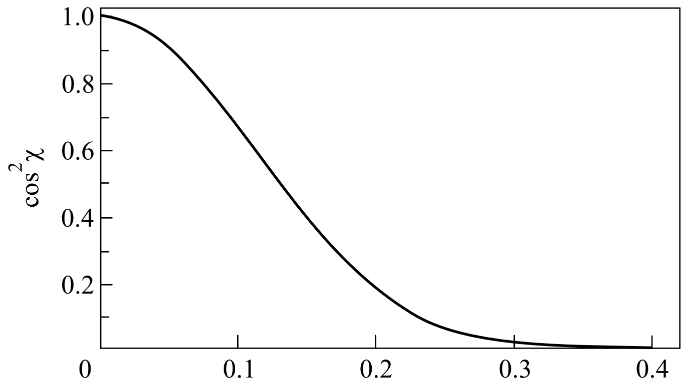

Fig.1. The square of the cosine of the observable angle between two parallel transported vectors along the geodesic $\gamma(0,\pi/2)$ and deformed geodesic $f(0,\pi/2)$ against the angle length in radians of the geodesic. The relationship between frequencies is as follows: $\Omega = 10\pi$ radians/s, $\omega = \pi$ radians/s

Let me consider briefly the expected phase modulation shift, accumulated during the parallel transport of the velocities of the ellipticity and the inclination angle in the real experiment described in the Section 1. The key role belongs to the curvature κ which I put equal to 1 in previous formulas. Now I assume that curvature of the state space $CP(N-1)$ is the measure of the correlations between the different LDV. Volkov *et al* used the sphere curvature as a phenomenological constant of the spin waves interaction [4]. I put the curvature as the fine structure constant $\kappa = e^2/\hbar c \approx 0.007$. The reason for this choice will be discussed elsewhere. If the modulation frequency $\Omega = 4000\pi$ radians/s has the angle amplitude $\beta = 0.017$ radians and the dragging frequency $\omega = 10\pi$ radians/s, the behavior of velocities of the ellipticity and the azimuth angle is shown in the Figs.2,3.

All LDV discussed above, say the velocity of ellipticity ϵ variation are measurable. Since now $\frac{d\epsilon}{dt}$ is curvature-dependent, it differs from the "flat" parallel transport. Then the instant frequency (the speed of the modulation phase variation) is the function of the ϵ and θ and it should contain besides the frequencies ω and $\beta\Omega$ the frequency $\kappa\beta\Omega$. It would be interesting to measure it in some experiment. The modulation frequency to this aim should be essentially higher than I used in my calculations.

The topological character of the Berry phase [8, 9], Aharonov-Anandan [10, 11] and the Wilczek-Zee phase

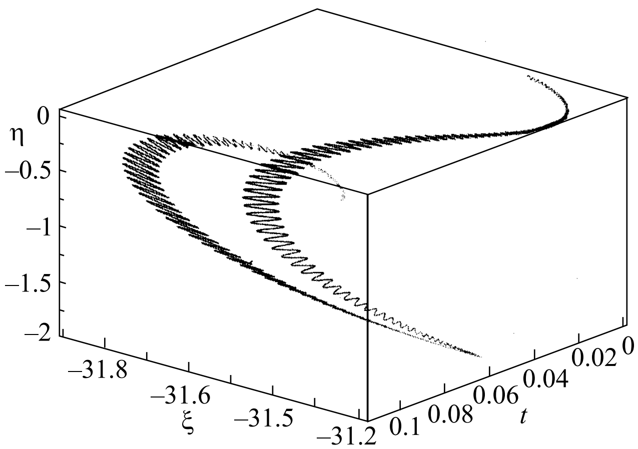

Fig.2. The time dependence of the ellipticity velocity. Initial conditions: $\xi = \Re(d\epsilon/dt) = -31.4$ radians/s, $\eta = \Im(d\epsilon/dt) = 0$

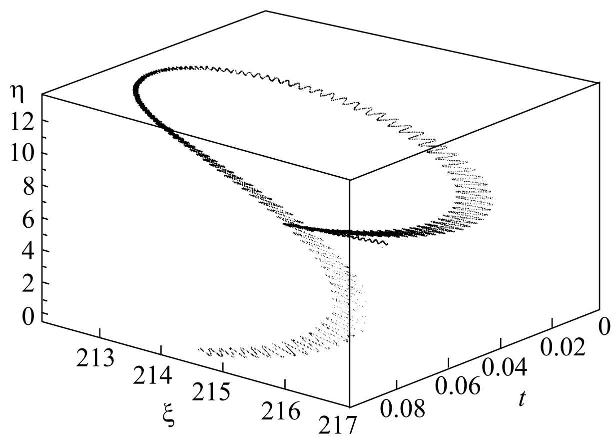

Fig.3. The time dependence of the azimuth angle velocity. Initial conditions: $\xi = \Re(d\theta/dt) = 213.5$ radians/s, $\eta = \Im(d\theta/dt) = 0$

[16] arise as a macroscopic environment reaction on the quantum dynamics of an "immersed" quantum system. The anholonomies of the "parallel transport" of the state vector are expressed as some effective gauge fields reflecting the topological character of the transformation groups of orientations of macroscopic elements (polarizers, $\lambda/4$ plates, etc.) of the quantum setup. Therefore it is not so strange that there are close classical analogies of the topological phases in classical physics (e.g. Hannay angle [9]). This is the reason why the dynamic phase should be discarded in order to get a definite geometric (topological) phase. Therefore, in general it is impossible of course, to define these gauge fields in some fundamental sense. But such gauge fields may by really fundamental in two important cases of the complex projective state space $CP(N-1)$. Firstly, since we believe that rays of quantum states are the fundamental notions at any level. Secondly, $CP(1)$ may be treated as the Qubit coherent state space under quantum information processing. In these cases there arises a new geometrodynamics phase which relates to the affine gauge field. Corresponding gauge fields associated with the curvature of $CP(N-1)$ are state-dependent and they realize the local gauge transformation of the moving quantum frame in $CP(N-1)$ [1, 15, 13, 17]. They are akin to the Wilczek-Shapere gauge fields related to the problem of a deformable body in fluid [18].

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