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One mode nonlinear regime of Weibel instability in a plasma with anisotropic temperature

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An analytical solution is found to the vortex electron anisotropic hydrodynamic equations which describe the nonlinear evolution of the long wavelength Weibel instability. The presented analytical approach shows that the long wavelength Weibel instability saturates without a decrease in the temperature anisotropy in the one mode regime due to the rotation of the anisotropy axes. The generated magnetic field is circular polarized and its amplitude varies periodically in time.

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It is well known that a plasma with an electron energy anisotropy is unstable [1]. The Weibel instability is a pure electromagnetic mode which describes magnetic field excitation with a characteristic time scale which is much longer than the electron plasma wave period. Such an instability likely appears in the interaction of a short duration X-ray pulse with a gas media where weak collisions are unable to rapidly remove the electron energy anisotropy arising from photoionization. In the past, the only known artificial source of short intense X-rays was a nuclear explosion. However, recent developments in intense ultrashort X-ray laser technology have opened a new regime of X-ray-matter interaction, in which intense X-ray laser pulses penetrate deep into the targets and produce nonequilibrium plasma of macroscopic size. The example is the XFEL project [2] which is generating interest in the need to understand extreme plasma states that can be encountered in interaction of X-rays with matter.

The most advanced studies dealing with the nonlinear stage of a plasma with an anisotropic temperature have been performed with numerical simulations [3 – 8]. It also is known that the vortex electron anisotropic hydrodynamics (VEAH) model [9] is well suited for analytical study of this problem. However, analytical results obtained to date mainly demonstrate the existence of explosion-like or self-similar solutions to the VEAH equations [9, 10], while the conditions of their approach with an initial value problem remain unclear. Recent three-dimensional (3D) particle-in-cell simulations (PIC) have shown that the Weibel instability for high temperature anisotropy ($T_\perp/T_\parallel \gg 1$) evolves to a one-mode 1D regime with a finite saturated averaged anisotropy ($T_\perp/T_\parallel \sim 1$) and a circularly polarized long scale length magnetic field [8]. The initial conditions for these simulations corresponded to a small amplitude stochastic initial magnetic field that is different from 1D VEAH numerical model for a one component magnetic field [7] where the single mode regime was found to exist for a short wavelength Weibel instability growing from the initial harmonic perturbations. The single mode regime helical polarization of the magnetic field which saturates due to periodic energy exchange with thermal electrons has been predicted in Ref.[11].

For a plasma with anisotropic temperature we examine the nonlinear evolution of a long wavelength Weibel instability, $ck < \omega_p$, where k is the wave number, ω_p is the electron plasma frequency, and c is the speed of light. We have found an exact one-mode solution of the VEAH equations which can be applied to the initial value problem for the Weibel instability itself and to provide analytic confirmation of the existence of the 1D saturation regime displayed in recent 3D PIC simulations [8].

The VEAH model [9] is formulated in terms of the coupled equations for the temperature tensor $\mathbf{T}$ and the quasi-static magnetic field $\mathbf{B}$. In the long wavelength limit, these equations read as follows

$$\frac{\partial \mathbf{\Omega}}{\partial t} = -\frac{1}{m_e}\nabla \times \nabla \cdot \mathbf{T}, \quad \frac{\partial \mathbf{T}}{\partial t} = \{\mathbf{T} \times \mathbf{\Omega}\}, \qquad (1)$$

where $\mathbf{\Omega} = eB/m_e c$ is the electron hyrofrequency, e and m_e are the electron charge and mass, and $\{...\}$ signifies a symmetrization of the tensor, $\{A_{ij}\} = A_{ij} + A_{ji}$. Eq. (1) conserve the temperature anisotropy. For a single-axis temperature anisotropy,

$$\mathbf{T} = T_\parallel \mathbf{nn} + T_\perp(\mathbf{I} - \mathbf{nn}), \qquad (2)$$

the longitudinal and transversal temperatures, $T_\parallel$ and $T_\perp$, do not change in time. Here $\mathbf{I}$ is the absolute unit tensor and $\mathbf{n}(\mathbf{r},t)$ is the unit vector in the direction of the anisotropy axis. We assume homogeneous $T_\parallel$ and $T_\perp$ in space, that corresponds to the standard Weibel instability [1].

The hypothesis that nonlinear regime of the long wavelength Weibel instability can be described in terms of $\mathbf{\Omega}$ and $\mathbf{n}$ has been put forward in Ref. [12]. However, the corresponding equations,

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \frac{T_\parallel - T_\perp}{m_e} \nabla \times (\mathbf{n} \times \nabla \times \mathbf{n} - \mathbf{n} \nabla \cdot \mathbf{n}), \qquad \frac{\partial \mathbf{n}}{\partial \mathbf{t}} = \{\mathbf{n} \times \mathbf{\Omega}\}, \quad |\,\mathbf{n}\,| = \mathbf{1}, \tag{3}$$

have not been solved for the initial value problem. In order to solve the initial value problem for the one dimensional case $(\partial/\partial z)$ we take advantage of the following parameterization

$$\mathbf{\Omega} = \{\gamma_0\, a(t) \sin kz,\ -\gamma_0\, a(t) \cos kz,\ 0\}, \qquad \mathbf{n} = \{\sin \Phi(\mathbf{t})\ \cos \mathbf{kz},\ \sin \Phi(\mathbf{t})\ \sin \mathbf{kz},\ \cos \Phi(\mathbf{t})\}. \tag{4}$$

This corresponds to circularly polarized single mode magnetic field. Here, for convenience we have introduced the growth rate of the long wavelength Weibel instability, $\gamma_0 = k\sqrt{(T_\perp - T_\parallel)/m_e}$.

One may confirm through the direct substitution of Eqs. (4) into Eqs. (3) that the assumed structure satisfies the VEAH model. The corresponding equations for the dimensionless amplitude of the magnetic field a and the angle Φ, which defines the evolution of the anisotropy axis, read

$$2\ddot{\Phi} = \sin 2\Phi, \quad a = \dot{\Phi}. \tag{5}$$

The solution to these equations are given in quadratures as follows:

$$\gamma_0\, t = \int_{\Phi_0}^{\Phi} d\varphi [a_0^2 + \sin(\varphi + \Phi_0) \sin(\varphi - \Phi_0)]^{-1/2}, \qquad a^2 = a_0^2 + \sin(\Phi + \Phi_0) \sin(\Phi - \Phi_0), \tag{6}$$

where the two constants of integration $a(0) = a_0$ and $\Phi(0) = \Phi_0$ can be related to the initial amplitude of the magnetic field and the initial direction of anisotropy axis. This nonlinear solution may be expressed in terms of the elliptic integral of the first kind.

For the standard problem definition for Weibel instability one assumes $\Phi_0 = 0$ and $a_0 \ll 1$. Initially, the amplitude of the magnetic field increases exponentially with a growth rate γ_0. After that, the growth of the magnetic field slows, the magnetic field reaches a maximum, and then decreases to the initial value. This process is then repeated in a periodic fashion. During one magnetic field cycle the anisotropy direction changes to the opposite direction but returns to the starting value during the next cycle. The period of the magnetic field pulsations slowly decreases with a_0. Figure 1 shows the time dependencies of the dimensionless amplitude of the magnetic field and the z-component of the anisotropy vector, n_z, for $n_z(0) = 1$, where the time scale is given in γ_0^{-1} units. The solutions for $a(t)$ and n_z are the multi-instanton solution and multi-kink solution, respectively.

Considerable initial amplitudes a_0 (e.g., Fig.1c) is also of interest for the evaluation of the nonlinear behav-

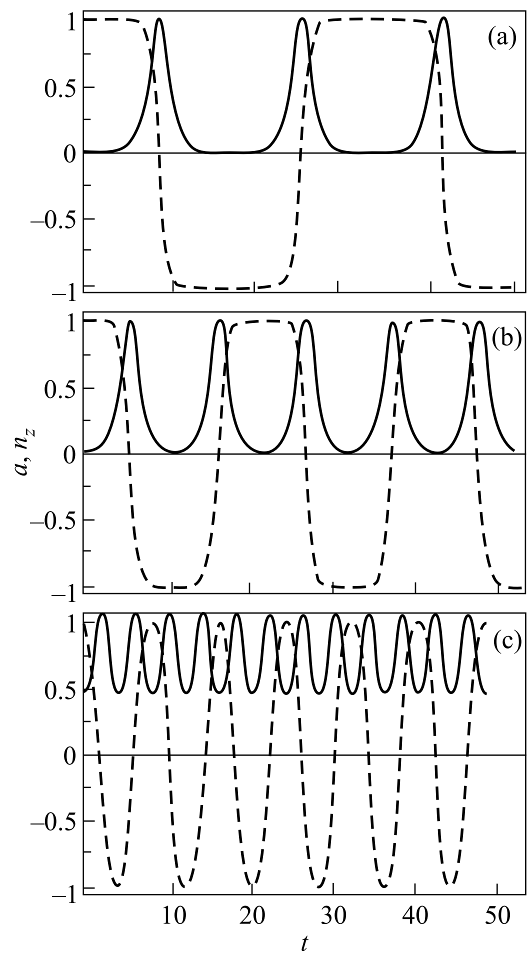

Fig.1. Evolution of the magnetic field (solid lines) and the anisotropy axis (dashed lines) for $\Phi_0 = 0$ and $a_0 = 0.001$ (a), 0.02 (b), and 0.5 (c)

iour of the Weibel instability which is initially excited in short wave length domain and then evolves to long scale lengths with the temperature anisotropy saturation and circular polarized magnetic field formation. Such behavior of the Weibel instability evolving to the 1D case was recently observed in 3D PIC simulations [8]. For these simulations, the analytic approach may be of particular interest because for the enlargement of the spacial scales the computations become very demanding. Corresponding comparison with the theory requires also to account for $\Phi(0) \neq 0$. In accordance with this, two qualitatively different situations are possible: $| a_0 | \lesseqgtr | \sin\Phi_0 |$. The VEAH solutions for $| a_0 | > | \sin\Phi_0 |$ are well represented by Fig.1 while the opposite case is illustrated by Fig.2. The specific case where $| a_0 | = | \sin\Phi_0 |$ is also

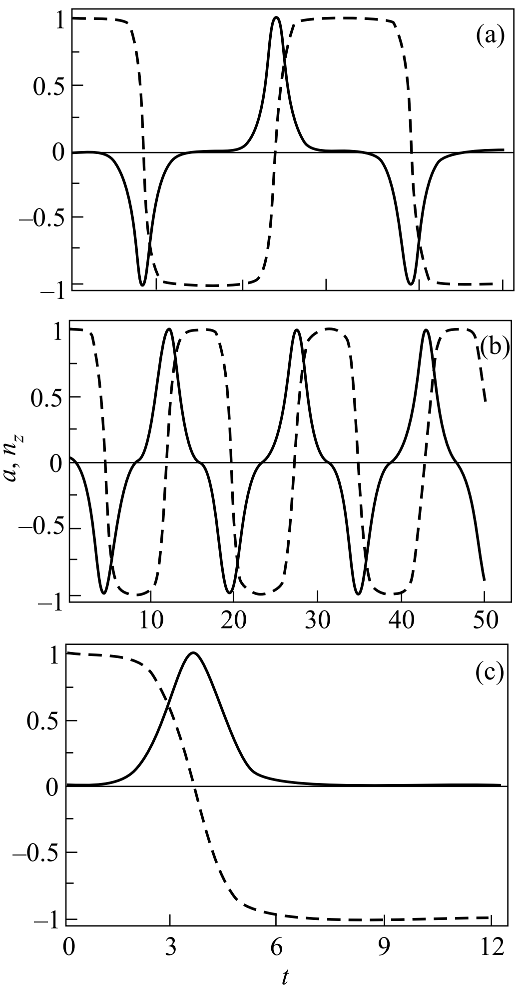

Fig.2. Evolution of the magnetic field (solid lines) and the anisotropy axis (dashed lines) for $a_0 = 0.001$, $\Phi_0 = 0.002$ (a), $a_0 = 0.05$, $\Phi_0 = 0.1$ (b), and $a_0 = \sin\Phi_0$, $\Phi_0 = 0.05$ (c)

presented (Fig.2c). The case where $| a_0 | < | \sin\Phi_0 |$ is represented by the bi-polar multi-instanton solution for the magnetic field amplitude. For the initial conditions $| a_0 | = | \sin\Phi_0 |$, the solutions for $a(t)$ and n_z become the solitary instanton and kink modes, respectively. However, as $t \to \infty$, the magnetic field and the inhomogeneity of the anisotropy for this solution disappear, i.e. it is unstable and Weibel instability must arise again.

The period, t_0, of the nonlinear solution (6) also depends on the sign of the expression $a_0^2 - \sin{\Phi_0}^2$. Denoting $p^2 \equiv a_0^2 - \sin{\Phi_0}^2$, one finds

$$t_0 = (4/\gamma_0\sqrt{1+p^2})\, K(1/\sqrt{1+p^2}) \qquad (7)$$

for the time required for a complete cycle of anisotropy rotation in the case $| a_0 | > | \sin\Phi_0 |$. Similarly, for $-p^2 \equiv a_0^2 - \sin{\Phi_0}^2 < 0$ this period reads

$$t_0 = (4/\gamma_0)\, K(\sqrt{1-p^2}), \qquad (8)$$

where $0 < p < 1$. In Eqs. (7) and (8), $K(u)$ is the complete elliptic integral of the first kind with the modulus u. In accordance with Figs.1 and 2, the period decreases with p (with the magnetic field amplitude in $\Phi_0 \to 0$ case), as shown in Fig.3, where the period is in units of

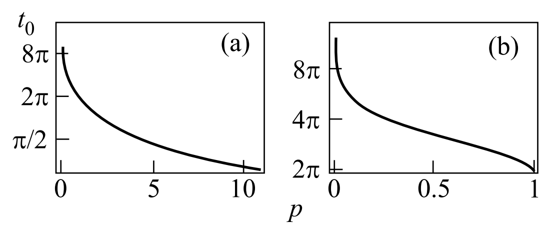

Fig.3. Period of the nonlinear solutions of VEAH for (a) $a_0^2 - \sin{\Phi_0}^2 > 0$ (logarithmic scale) and (b) $a_0^2 - \sin{\Phi_0}^2 < 0$

γ_0. The soliton solution corresponds to $p = 0$. As $p \to 1$, Eq. (8) gives the period $t_0 = 2\pi$. For example, in the case of the standard problem definition for the Weibel instability, $n_z(0) = 1$, the period of nonlinear pulsations is $t_0 \simeq 8\,\pi$ for $a_0 = 0.01$ and $t_0 \simeq 4\,\pi$ for $a_0 = 0.25$.

The averaged (over the period) magnetic energy is small, $\langle B^2\rangle/8\pi n_e T_\perp \ll 1$, because $ck < \omega_p$ and the duration of nonlinear pulsations is less than t_0 providing $\langle a^2\rangle < 1$, viz

$$\frac{\langle B^2\rangle}{8\pi n_e T_\perp} = \frac{\langle a^2\rangle}{2}\left(1 - \frac{T_\parallel}{T_\perp}\right)\frac{c^2k^2}{\omega_p^2}, \qquad (9)$$

where n_e is the electron density. This is illustrated by Fig.4 for $\Phi_0 = 0$, where the dimensionless magnetic energy $\langle a^2\rangle$ has very weak dependence on a_0. A small saturated magnetic field energy $\langle B^2\rangle/8\pi n_e T_\perp \sim 0.02$ has also been observed in PIC simulations [8], where in the final state, the one mode regime was observed with $ck/\omega_p \simeq 0.5$ and the saturated anisotropy $T_\parallel/T_\perp \simeq 1/3$. Assuming from Fig.4 a rank value $\langle a^2\rangle \sim 1/4$, one can estimate from Eq. (9) that $\langle B^2\rangle/8\pi n_e T_\perp$ is very close to the PIC simulation results [8].

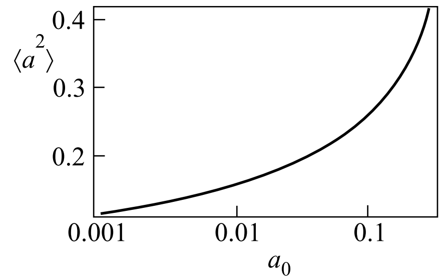

Fig.4. Averaged dimensionless magnetic field energy

In summary, we have obtained an analytical solution to the VEAH equations (3) which describe the relaxation of the long wavelength Weibel instability which originates from the anisotropy of the electron temperature. The mechanism responsible for the saturation of the instability is the rotation of the anisotropy axis rather than the temperature becoming isotropic. Such a rotation leads to dephasing of the generated magnetic field with respect to the source of the anisotropy. The solution that has been found corresponds to single mode circular polarized magnetic field. Our theoretical model explains the formation of single mode magnetic structures that are observed in PIC simulations [8]. The characteristics of the Weibel plasma have been derived and this can have a fundamental significance and potential application in the implementation of the forefront XFEL project [13].

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