Pis'ma v ZhETF, vol. 76, iss. 1, pp. 60 – 62

Energy bounds of linked vortex states

A. P. Protogenov[1)], *V. A. Verbus*

Institute of Applied Physics of the RAS, 603950 Nizhny Novgorod, Russia

Institute for Physics of Microstructures of the RAS, 603950 Nizhny Novgorod, Russia

Submitted 30 April 2002

Resubmitted 30 May 2002

Energy bounds of knotted and linked vortex states in a charged two-component system are considered. It is shown that a set of local minima of free energy contains new classes of universality. When the mutual linking number of vector order parameter vortex lines is less than the Hopf invariant, these states have lower-lying energies.

PACS: 74.20.De, 74.20.Mn, 11.15.Tk, 11.10.Lm, 11.27.+d

A tangle of vortex filaments is a system which attract attention due to several reasons. Along with the coherent state, which is the background of this vortex field distributions, the filament system also contains a disorder combination due to free motion of its fragments and a topological order because of the effects of knotting and linking [1, 2] of its separate parts.

The study of soft condensed matter, whose universal behavior is determined by topological characteristics, is recognized as one of the most challenging problems of modern condensed matter physics [3]. The aim of this paper is to find the energy bounds for vortex states with a set of numbers determining the knotting and linking degree of the fields that take part in the description of the coherent state. We will use the Ginzburg-Landau model

$$F = \int d^3x \left[\sum_\alpha \frac{1}{2m} \left| \left(\hbar\partial_k + i\frac{2e}{c}A_k \right) \Psi_\alpha \right|^2 + + \sum_\alpha \left(-b_\alpha|\Psi_\alpha|^2 + \frac{c_\alpha}{2}|\Psi_\alpha|^4 \right) + \frac{\mathbf{B}^2}{8\pi} \right] \quad (1)$$

with a two-component order parameter

$$\Psi_\alpha = \sqrt{2m}\,\rho\,\chi_\alpha, \quad \chi_\alpha = |\chi_\alpha| e^{i\varphi_\alpha}, \quad (2)$$

satisfying the CP^1 condition, $|\chi_1|^2 + |\chi_2|^2 = 1$. This model is used in the context of the two-gap superconductivity [4, 5] and in the non-Abelian field theory [6, 7] (see also [8]).

It has been shown in paper [4] that there exists an exact mapping of the model (1), (2) into the following version of $\mathbf{n}$-field model:

$$F = \int d^3x \left[\frac{1}{4}\rho^2 (\partial_k \mathbf{n})^2 + (\partial_k \rho)^2 + + \frac{1}{16}\rho^2 \mathbf{c}^2 + (F_{ik} - H_{ik})^2 + V(\rho, n_3) \right]. \quad (3)$$

To write down Eq. (3) dimensionless units and gauge invariant order parameter fields of the unit vector $\mathbf{n} = \bar{\chi}\boldsymbol{\sigma}\chi$, where $\bar{\chi} = (\chi_1^*, \chi_2^*)$, $\boldsymbol{\sigma}$ – Pauli matrices, and of the velocity $\mathbf{c} = \mathbf{J}/\rho^2$ have been used. The full current $\mathbf{J} = 2\rho^2(\mathbf{j} - 4\mathbf{A})$ has a paramagnetic ($\mathbf{j} = i[\chi_1 \nabla \chi_1^* - \text{c.c.} + (1 \to 2)]$) and a diamagnetic ($-4\mathbf{A}$) parts. Besides in Eq. (3) $F_{ik} = \partial_i c_k - \partial_k c_i$, $H_{ik} = \mathbf{n} \cdot [\partial_i \mathbf{n} \times \partial_i \mathbf{n}] \equiv \partial_i a_k - \partial_k a_i$.

Setting in Eq. (3) $\mathbf{c} = 0$ we get Faddeev-Niemi model [9]. The numerical study of the knotted configurations of $\mathbf{n}$-field in this model has been done in [10 – 12]. The lower energy bound in this case

$$F \geqslant 32\pi^2 |Q|^{3/4} \quad (4)$$

is determined [13 – 15] by the Hopf invariant,

$$Q = \frac{1}{16\pi^2} \int d^3x\, \varepsilon_{ikl} a_i \partial_k a_l. \quad (5)$$

At compactification $\mathbb{R}^3 \to S^3$ and $\mathbf{n} \in S^2$, the integer $Q \in \pi_3(S^2) = \mathbb{Z}$ shows the degree of linking or knotting of filamental manifolds, where the vector field $\mathbf{n}(x, y, z)$ is defined. In particular, for two linked rings (Hopf linking) $Q = 1$, for the trefoil knot $Q = 6$ and etc. Significant point is the following: $\pi_3(CP^M) = 0$ at $M > 1$ and $\pi_3(CP^1) = \pi_3(S^2) = \mathbb{Z}$ [16]. In the latter case the order parameter (2) is two-component one [4] and linked or knotted soliton configurations are labeled by the Hopf invariant (5). In the $(3+0)D$ case of the free energy (3), Hopf invariant (5) is analogous to the Chern- Simons action $(k/4\pi) \int dt\, d^2x\, \varepsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$ determining strong correlations of $(2+1)D$ modes [17, 18] at semion value $k \simeq 2$.

[1)]e-mail: alprot@appl.sci-nnov.ru

Let us assume that ρ can be find from the minimal value of the potential $V(\rho)$ and the velocity $\mathbf{c}$ does not equal zero. Equation(3) in this case has the following form:

$$F = F_n + F_c - F_{int} = \int d^3x \times$$
$$\times \left[\left((\partial_k \mathbf{n})^2 + H_{ik}^2 \right) + \left(\frac{1}{4}\mathbf{c}^2 + F_{ik}^2 \right) - 2F_{ik}H_{ik} \right]. \quad (6)$$

It is seen from Eq. (6) that a superconducting state with $\mathbf{c} \neq 0$ has the energy which is less than the minimum in Eq. (4) due to renormalization of the coefficient ($= 1$) in the second term of the functional F_n. To find the lower free energy bound in the superconducting state with $\mathbf{c} \neq 0$ we will use the following inequality:

$$F_n^{5/6}\, F_c^{1/2} \geqslant (32\pi^2)^{4/3}\, |L|, \quad (7)$$

where

$$L = \frac{1}{16\pi^2} \int d^3x\, \varepsilon_{ikl} c_i \partial_k a_l \quad (8)$$

is the degree of mutual linking [19, 20] of the velocity $\mathbf{c}$ lines and of the magnetic field $\mathbf{H} = [\nabla \times \mathbf{a}]$ lines. It is also an integral of motion [20, 21].

The proof of the inequality (7) employs the following set of inequalities:

$$|L| \leqslant ||\mathbf{c}||_6 \cdot ||\mathbf{H}||_{6/5} \leqslant 6^{1/6}||[\nabla \times \mathbf{c}]||_2 \cdot ||\mathbf{H}||_{6/5} \leqslant$$
$$\leqslant 6^{1/6}||[\nabla \times \mathbf{c}]||_2 \cdot ||\mathbf{H}||_1^{2/3} \cdot ||\mathbf{H}||_2^{1/3} \leqslant$$
$$\leqslant (32\pi^2)^{-4/3} F_c^{1/2} \cdot F_n^{2/3} \cdot F_n^{1/6} =$$
$$= (32\pi^2)^{-4/3} F_n^{5/6} \cdot F_c^{1/2}. \quad (9)$$

Here $||\mathbf{H}||_p \equiv (\int d^3x\, |\mathbf{H}|^p)^{1/p}$. At the first and third steps, we used the Hölder inequality $||\mathbf{fg}|| \leqslant ||\mathbf{f}||_p \cdot ||\mathbf{g}||_q$ where $1/p + 1/q = 1$. Under the condition $\nabla \cdot \mathbf{c} = 0$, the second step corresponds to the use of the Ladyzhenskaya inequality [22] $||\mathbf{c}||_6 \leqslant 6^{1/6}||[\nabla \times \mathbf{c}]||_2$. The fourth step in Eq. (9) arises after comparison of the terms $||\nabla \times \mathbf{c}||$ and $||\mathbf{H}||$ with the terms F_c and F_n in Eq. (6). The coefficient in Eq. (9) in our case turns out to be the same as in Eq. (4) [15]. The last line in Eq. (9) has shown also the contributions from $\mathbf{n}$- and $\mathbf{c}$-parts of free energy (6) to the final result (7).

Applying the Schwartz-Cauchy-Bunyakovsky inequality to F_{int} in Eq. (6) yields

$$F_{int} \leqslant 2||F_{ik}||_2 \cdot ||H_{ik}||_2 \leqslant 2F_c^{1/2} \cdot F_n^{1/2}. \quad (10)$$

The equality in the r.h.s. of Eq. (10) is achieved in the limit of a small size of linked vortex configurations. Substituting the boundary value of F_{int} into Eq. (6) we get

$$F \geqslant F_{min} = F_n \left(1 - \sqrt{F_c/F_n}\right)^2. \quad (11)$$

The Hopf configuration with $Q = 1$ for which the equality in Eq. (4) is reached, is two linked rings of the radius R and $(F_n)_{min} = 2\pi^2 R^3 (8/R^2 + 8/R^4)|_{R=1} = 32\pi^2$. Accordingly in our case $\mathbf{c} \neq 0$ we will assume the existence of configurations satisfying the equality in Eq. (7). It is important to emphasize that for small ρ in Eq. (3) and therefore for large $\mathbf{c}$ (because all terms in Eq. (3) are of the same order), we encounter the instability of linked vortex with respect to dilatation transformations [15]. This results in the restriction of F_c from above. Taking into account this remark we will use for F_c in Eq. (11) the lower bound $F_c^{1/2} = (32\pi)^{4/3} F_n^{-5/6}|L|$ in Eq. (7) and $F_n = 32\pi^2|Q|^{3/4}$ to have finally

$$F \geqslant 32\pi^2\, |Q|^{3/4}\, (1 - |L|/|Q|)^2. \quad (12)$$

The inequality (12) is the main result of the paper. The trivial case $Q = 0$ should be considered after the limit $L = 0$. Let us pay also attention to the self-dual relation $F_n = \alpha F_c$ with $\alpha \sim 1$ which follows from F_{min}.

It follows from Eq. (12) that for all numbers $L < Q$ the energy of the ground state is less than that in the model described in [9], for which the inequality (4) is valid. The origin of the energy decrease can be easily understood. Even under the conditions of the existence of the paramagnetic part $\mathbf{j}$ of the current $\mathbf{J}$, the diamagnetic interaction in the superconducting state consumes its own current energy and a part of the energy relating to the $\mathbf{n}$-field dynamics for all state classes with $L < Q$.

The case when ρ is not constant both for $\mathbf{c} = 0$ [23] and $\mathbf{c} \neq 0$ is of a certain interest. It is more complicated due to some reasons and will be considered in a separate paper (see also [24]). We only mention that in a soft case $\rho \neq$ const we need to have the proof of the compatibility of the peculiar value of the coefficient in the r.h.s of Eq. (7) with stability condition [15]. The equality in Eq. (12) under this remark should be understood as an ideal limit depending on topologial characteristics of knots and links only.

In conclusion, we have found the energy bounds of the superconducting states using CP^1 version of Ginzburg-Landau model under the conditions of the existence of linking and knotting phenomena of the $\mathbf{n}$- and $\mathbf{c}$-fields being the gauge invariant order parameters of the considered system. We have shown that the energy space of the local minima contains new state classes with $L < Q$.

We would like to thank A. G. Abanov, L. D. Faddeev, E. A. Kuznetsov and G. E. Volovik for advices, V. F. Gantmakher for the crucial remark, and G. M. Fraiman, A. G. Litvak, V. A. Mironov for useful discussions. This work was supported in part by the RFBR under the grant # 01-02-17225.

1. V. M. H. Ruutu, U. Parts, J. H. Koivuniemi et al., JETP Lett. **60**, 671 (1994).
2. Yu. G. Makhlin and T. Sh. Misirpashaev, JETP Lett. **61**, 49 (1995).
3. V. Katritch, J. Bednar, D. Michoud et al., Nature **384** 142 (1996).
4. E. Babaev, L. D. Faddeev, and A. J. Niemi, Phys. Rev. **B65**, 100512 (2002); cond-mat/0009438.
5. E. Babaev, Phys. Rev. Lett. **88**, 177002 (2002).
6. L. D. Faddeev and A. J. Niemi, Phys. Lett. **B525**, 195 (2002).
7. Y. M. Cho, Phys. Rev. Lett. **87**, 252001 (2001); hep-th/0110076.
8. P. van Baal and A. Wipf, hep-th/0105141.
9. L. D. Faddeev and A. J. Niemi, Nature **387**, 58 (1997).
10. J. Gladikowski and M. Hellmund, Phys. Rev. **D56**, 5194 (1997).
11. R. A. Battye and P. M. Sutcliffe, Phys. Rev. Lett. **81**, 4798 (1998).
12. J. Hietarinta and P. Salo, Phys. Lett. **B451**, 60 (1999).
13. A. F. Vakulenko and L. V. Kapitansky, Sov. Phys. Dokl. **24**, 433 (1979).
14. A. Kundu and Yu. P. Rubakov, J. Phys. **A15**, 269 (1982).
15. R. S. Ward, Nonlinearity **12**, 1 (1999), hep-th/9811176.
16. A. G. Abanov and P. W. Wiegmann, hep-th/0105213.
17. L. A. Abramyan, A. P. Protogenov, and V. A. Verbus, JETP Lett. **69**, 887 (1999).
18. A. P. Protogenov, JETP Lett. **73**, 255 (2001).
19. V. I. Arnold and B. A. Khesin *Topological Methods in Hydrodynamics*, Appl. Math. Sci. **125**, Chapt. 3.
20. V. E. Zakharov and E. A. Kuznetsov, Phys. Usp. **40**, 1087 (1997).
21. H. K. Moffatt, J. Fluid Mech. **106**, 117 (1969).
22. O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Gordon and Breach, 1969.
23. M. Lübcke, S. M. Nasir, A. Niemi, and K. Torokoff, hep-th/0106102.
24. A. P. Protogenov, cond-mat/0205133.