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Soft and hard processes in QCD

I. M. Dremin[1)]

Lebedev Physical Institute RAS, 119991 Moscow, Russia

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QCD equations for the generating functions are applied to separate soft and hard jets in e^+e^--processes of multiparticle production. The dependence of average multiplicities and higher moments of multiplicity distributions of particles created in a "newly born" soft subjets on the share of energy devoted to them is calculated in fixed coupling gluodynamics. This dependence is the same as for the total multiplicity up to a constant factor if soft jets are defined as those carrying out a fixed share of initial energy at all energies. The constant factor depends on this share in a non-trivial way. Other definitions are also proposed. The relation between these quantities for soft and hard processes is discussed.

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In multiparticle production, it is quite common procedure to separate all processes into the soft and hard ones. Even though the intuitive approach is appealing, the criteria of the separation differ. It is shown below that QCD equations for the generating functions can be applied to this problem. It is demonstrated how the average multiplicities of soft and hard processes depend on the parameter which is used to distinguish them. The same method can be applied to any moment of the multiplicity distributions as is explicitely shown for the second moment (dispersion).

The QCD equations for the generating functions (functionals) are known since long ago (e.g., see the book [1]). This is the system of two integro-differential equations which describe the quark and gluon jets evolution. They are quite useful for prediction and description of many properties of high energy jets (for the reviews see, e.g., [2 – 4]). It has been found that the main qualitative features of the process can be safely predicted by considering the single equation for gluon jets evolution. In that way one neglects quarks and treats the gluodynamics in place of the chromodynamics. Moreover, its solution may be further simplified if one disregards the running property of the QCD coupling strength and considers it as a fixed one (see papers [5]). To avoid some technicalities, we adopt this approach in what follows and treat the multiplicity distributions of gluon jets. Both quark and gluon jets with running coupling strength will be considered in the QCD context elsewhere.

When an initial gluon splits into two gluons (subjets), its energy E is additively shared among them, and the multiplicity of the whole process is a sum of multiplicities of these two subjets. The energy dependence of mean multiplicity of particles created in a subjet, which carries out some share of initial energy xE with a fixed value of x, must be the same as for the initial jet if gluons are equivalent. In experiment it is more convenient to deal with values of x ranging in some finite interval to get enough statistics. One of the ways is to separate all subjets into soft and hard ones if the parameter x is smaller or larger than some x_0. We show how properties of these two sets behave with energy. We shall also consider the case when the parameter x_0 depends on initial energy.

If the probability to create n particles[2)] in a jet is denoted as P_n, the generating function G is defined as

$$G(z,y)=\sum_{n=0}^{\infty}P_n(y)(1+z)^n, \qquad (1)$$

where z is an auxiliary variable, $y=\ln(p\Theta/Q_0)=\ln(2Q/Q_0)$ is the evolution parameter, defining the energy scale, p is the initial momentum, Θ is the angle of the divergence of the jet (jet opening angle), assumed here to be fixed, Q is the jet virtuality, $Q_0=\text{const}$.

The gluodynamics equation for the generating function is written as

$$\frac{dG}{dy}=\int_0^1 dxK(x)\gamma_0^2[G(y+\ln x)G(y+\ln(1-x))-G(y)], \qquad (2)$$

where

$$\gamma_0^2=6\alpha_S/\pi, \qquad (3)$$

[1)]e-mail: dremin@lpi.ru

[2)]In what follows, we adopt the local parton-hadron duality hypothesis with no difference between the notions of particles and partons up to some irrelevant factor.

α_S is the coupling strength and the kernel $K(x)$ is

$$K(x) = 1/x - (1-x)[2 - x(1-x)]. \quad (4)$$

One should not be surprised that the shares of energy x and $1-x$ devoted to two gluons after the initial one splits to them enter asymmetrically in this equation. Surely, the initial equation is fully symmetrical. The asymmetry is introduced when the phase space is separated in two equally contributing parts and one of the jets with the share x is called as a "newly born" one (for more details see [1]). Therefore we shall call soft processes those where soft newly born jets are produced, i.e. those where x is small enough ($x \le x_0 \ll 1$). In e^+e^--experiments, this would correspond to considering soft newly born gluon jets with energies $E_g \le x_0 E \ll E$ in 3-jet events.

Before separating soft and hard jets, let us stress that, at a given energy, this is an additive procedure for probabilities $P_n = P_{ns} + P_{nh}$ and, consequently, for $G = G_s + G_h$, where indices s and h are for soft and hard processes, correspondingly. It is convenient to rewrite the generating function in terms of unnormalized factorial moments

$$\mathcal{F}_q = \sum_n P_n n(n-1)...(n-q+1) = \frac{d^q G(z)}{dz^q}\Big|_{z=0}, \quad (5)$$

so that

$$G = \sum_{q=0}^{\infty} \frac{z^q}{q!}\mathcal{F}_q. \quad (6)$$

The low rank moments are

$$\mathcal{F}_1 = \langle n\rangle, \quad \mathcal{F}_2 = \langle n(n-1)\rangle = D^2 + \langle n\rangle^2 - \langle n\rangle \quad (7)$$

and D is the dispersion

$$D^2 = \langle n^2\rangle - \langle n\rangle^2. \quad (8)$$

It is seen from Eq. (5) that unnormalized moments are additive also. To retain the additivity property we define the normalized factorial moments for soft and hard jets with the normalization to the total mean multiplicity but not to their multiplicities

$$F_q = \frac{\mathcal{F}_q}{\langle n\rangle^q} = \frac{\mathcal{F}_{qs} + \mathcal{F}_{qh}}{\langle n\rangle^q} = F_{qs} + F_{qh}. \quad (9)$$

Thus the total multiplicity is in the denominator. The additivity would be lost if soft and hard values are normalized to their average multiplicities $\langle n_s\rangle$ and $\langle n_h\rangle$. Introducing $f_q = F_q/q!$ we write

$$G = \sum_{q=0}^{\infty} z^q \langle n\rangle^q f_q. \quad (10)$$

The scaling property of the fixed coupling QCD [5] allows to look for the solution of the equation (2) with

$$\langle n\rangle \propto \exp(\gamma y) \quad (\gamma = \text{const}) \quad (11)$$

and get the system of iterative equations for f_q

$$\gamma q f_q = \gamma_0^2 \int_0^1 dx K(x)[(x^{\gamma q} + (1-x)^{\gamma q} - 1) f_q + \\ + \sum_{m=1}^{q-1} x^{\gamma m}(1-x)^{\gamma(q-m)} f_m f_{q-m}]. \quad (12)$$

By definition $f_1 = 1$ and one gets at $q = 1$ the relation between γ and γ_0

$$\gamma = \gamma_0^2 \int_0^1 dx K(x)(x^\gamma + (1-x)^\gamma - 1) = \gamma_0^2 M_1(1,\gamma), \quad (13)$$

where

$$M_1(z,\gamma) = \int_0^z dx K(x)(x^\gamma + (1-x)^\gamma - 1). \quad (14)$$

Let us point out that Eq. (13) is derived from the equation for mean multiplicities which follows from Eq. (2):

$$\langle n(y)\rangle' = \int_0^1 dx \gamma_0^2 K(x)(\langle n(y + \ln x)\rangle + \\ + \langle n(y + \ln(1-x))\rangle - \langle n(y)\rangle). \quad (15)$$

These results are well known [5]. Here, we would like to consider Eq. (13) in more detail. As follows from Eq. (15), the first two terms in the brackets correspond to mean multiplicities of two subjets, and their sum is larger than the third term denoting the mean multiplicity of the initial jet (all divided by E^γ). Therefore, the integrand is positive, and Eq. (13) defines the anomalous dimension γ. This does not contradict to the statement that for a given event the total multiplicity is a sum of multiplicities in the two subjets because the averages in Eq. (15) are done at different energies.

For small enough γ and γ_0 one gets

$$\gamma \approx \gamma_0(1 - 0.458\gamma_0 + 0.213\gamma_0^2). \quad (16)$$

For the second moment one gets from Eq. (12) at $q = 2$ for small γ:

$$F_2 \approx \frac{4}{3}(1 - 0.31\gamma). \quad (17)$$

Now, according to the above discussion we define soft jets as those with sum of energies of belonging to them particles less than some $x_0 E$. First, consider x_0=const and small. Then we should choose the upper limit of integration in eq. (12) equal to x_0. Therefore the moments of soft processes f_{qs} are calculated as

$$\gamma q f_{qs} = \gamma_0^2 \int_0^{x_0} dx K(x)[(x^{\gamma q} + (1-x)^{\gamma q} - 1) f_q + \\ + \sum_{m=1}^{q-1} x^{\gamma m}(1-x)^{\gamma(q-m)} f_m f_{q-m}]. \quad (18)$$

One should not be confused that the total moments (obtained from the average multiplicities of both soft and hard jets) are in the integrand of Eq. (18). This is related to the difference between the notions of multiplicity in a given event and their averages discussed above. The integration over small x up to x_0 chooses just the mean multiplicity of particles belonging to soft jets $\langle n_s \rangle$ while the integration from x_0 to 1 gives that for hard jets.

For $q = 1$ one gets from (18)

$$\frac{\langle n_s \rangle}{\langle n \rangle} = \frac{M_1(x_0, \gamma)}{M_1(1, \gamma)}. \quad (19)$$

For small x_0 it is

$$\frac{\langle n_s \rangle}{\langle n \rangle} \approx \frac{\gamma_0^2}{\gamma^2} x_0^\gamma N_1(x_0, \gamma), \quad (20)$$

$$N_1(x_0, \gamma) = 1 - \gamma^2 x_0^{1-\gamma} - \frac{2\gamma}{1+\gamma} x_0 + \\ + \frac{\gamma^2(3+\gamma)}{4} x_0^{2-\gamma} + \frac{3\gamma}{2+\gamma} x_0^2 - \frac{\gamma^2(2+\gamma)}{3} x_0^{3-\gamma}. \quad (21)$$

Thus we have found the energy dependence of mean multiplicity of particles in a set of subjets with low energies $E_s \le x_0 E$. As expected for constant x_0, it is the same as the energy dependence of the total multiplicity with a different factor in front of it. Namely this dependence should be checked first in experimental data. Imposed on one another, these figures should coincide up to a normalization factor (19). This would confirm universality of gluons in jets.

Quite interesting is the non-trivial dependence of the normalization factor in Eq. (19) on the parameter x_0, which does not coincide simply with x_0^γ. It reflects the structure of QCD kernel $K(x)$. The main dependence on the cut-off parameter x_0 is given for $x_0 \ll 1$ by the factor x_0^γ with the same power as in dependence of total multiplicity on energy. This corresponds to subjets with the largest energy of the set. However, with increase of x_0, this dependence is modified according to Eqs. (19)-(21). The negative corrections become more important in Eq. (21). They are induced by subjets with energies lower than $x_0 E$. Theit shape reflects the fact that these subjets are weighted according to the kernel $K(x)$ determined by the QCD lagrangian. The decrease of the normalization factor corresponds to diminishing role of very low energy jets at higher initial energies. This should be also checked in experiment.

If plotted as a function of the maximum energy in a set of jets ϵ_m, the mean multiplicity is

$$\langle n_s \rangle \propto \epsilon_m^\gamma \left[1 - \gamma^2 \left(\frac{\epsilon_m}{E}\right)^{1-\gamma} - \frac{2\gamma}{1+\gamma}\left(\frac{\epsilon_m}{E}\right) + \right. \\ + \frac{\gamma^2(3+\gamma)}{4}\left(\frac{\epsilon_m}{E}\right)^{2-\gamma} + \frac{3\gamma}{2+\gamma}\left(\frac{\epsilon_m}{E}\right)^2 - \\ \left. - \frac{\gamma^2(2+\gamma)}{3}\left(\frac{\epsilon_m}{E}\right)^{3-\gamma}\right]. \quad (22)$$

It reminds Eq. (11) with the correction factor in the brackets (for $\epsilon_m \ll E$ as in (21)).

This is the consequence of the scaling property of the fixed coupling QCD which results in the jets selfsimilarity. The relative weights of soft and hard processes are determined by the factor γ_0^2/γ^2 as seen from Eqs (20), (21). They can be used to find out this ratio in experiment.

For $q = 2$ we obtain

$$F_{2s} = [0.5 F_2 M_1(x_0, 2\gamma) + M_2(x_0, \gamma)]/M_1(1, \gamma), \quad (23)$$

where

$$M_2(z, \gamma) = \int_0^z dx K(x) x^\gamma (1-x)^\gamma. \quad (24)$$

For small x_0 one gets

$$F_{2s} = \frac{\gamma_0^2}{\gamma^2} x_0^\gamma [0.25 F_2 x_0^\gamma N_1(x_0, 2\gamma) + 2N_2(x_0, \gamma)], \quad (25)$$

$$N_2(x_0, \gamma) = 1 - \gamma\frac{2+\gamma}{1+\gamma} x_0 + \gamma\frac{6+3\gamma+\gamma^2}{2(2+\gamma)} x_0^2. \quad (26)$$

Again, the main dependence on the cut-off parameter x_0 is provided by the factor x_0^γ.

Using these equations we have calculated the mean multiplicities and second moments of multiplicity distributions for soft jets. They are shown in Table for different choices of γ and γ_0 considered as the most realistic ones in previous studies. Note that $M_1(z, 2\gamma) = 0$ for $\gamma = 0.5$ (and so is $N_1(x_0, 2\gamma)$). One can notice that at larger x_0 the values in Table decline from x_0^γ-behaviour in accordance with eqs (19), (23).

The values for hard jets are obtained by subtracting these results from values for the total process. Small values of the second factorial moments do not imply that multiplicity distributions in soft jets are sub-poissonian because they are normalized to the total mean multiplicity. To get the genuine second factorial moments for

The values of mean multiplicity and second normalized factorial moment for different values of the coupling strength and cut-off parameters x_0

	$\gamma = 0.5$, $\gamma_0 = 0.7$		$\gamma = 0.4$, $\gamma_0 = 0.516$		$\gamma = 0.3$, $\gamma_0 = 0.36$	
x_0	n_s/n	F_{2s}	n_s/n	F_{2s}	n_s/n	F_{2s}
0.1	0.543	0.39	0.605	0.66	0.680	0.76
0.2	0.702	0.51	0.746	0.83	0.798	0.91
0.3	0.798	0.59	0.829	0.93	0.865	1.00

these processes one should divide the numbers in F_{2s}-columns to squared values in n_s/n-columns. In this way one gets quite large numbers so that these processes are super-poissonian but note that the genuine moments are not additive anymore. However, the statement about the widths of the distributions can be confronted to experimental data as well.

In principle, other definitions of soft jets are possible with $x_0 = x_0(E)$. Then one should solve the equation

$$\frac{d\langle n_s\rangle}{dE} = E^{\gamma-1}\gamma_0^2 M_1(x_0(E),\gamma), \qquad (27)$$

which follows from eq. (15). For example, one can choose the jets with energies less than some fixed constant independent of the initial energy. This would imply ϵ_m =const or $x_0(E) \propto 1/E$, and the exact integration of Eq. (27) is necessary. However, for qualitative estimates, Eqs (20)–(22) can be used. They show that the average multiplicity tends to a constant at high energies corresponding to the multiplicity at the upper limit. At lower energies, it slightly increases with energy due to increasing role of jets with energies closest to their upper limit.

It is well known that for running coupling the power dependence $s^{\gamma/2}$ is replaced by $\exp(c\sqrt{\ln s})$. The qualitative statement about the similar energy behaviour of mean multiplicities in soft and inclusive processes should be valid also.

The above results can be confronted to experimental data if soft jets are separated in 3-jet events. However, in our treatment we did not consider the common experimental cut-off which must be also taken into account. This is the low-energy cut-off imposed on a soft jet for the third jet to be observable. It requires the soft jet not to be extremely soft. Otherwise the third jet is not separated and the whole event is considered as a 2-jet one. Thus the share of energy must be larger than some x_1, and the integration in Eq. (14) should be from x_1 to x_0. For $x_1 \leq x_0 \ll 1$ one gets

$$\frac{\langle n_s\rangle}{\langle n\rangle} = \frac{\gamma_0^2}{\gamma^2}[v(x_0) - v(x_1)], \qquad (28)$$

where the function $v(x)$ is easily guessed from Eqs (20), (21). At $x_1 \ll x_0 \ll 1$ Eq. (20) is restored.

The cumulant moments of the distribution are not additive because they are obtained as derivatives of the generating function logarithm which is not additive for additive G. Thus H_q-moments [6, 7] are not additive also. Nevertheless, the role of hard jets can be traced by the preasymptotical oscillations of H_q. These oscillations are induced by the terms of the kernel K additive to the $1/x$-term. Thus the oscillations of H_q are the sensitive test of the shape of non-infrared terms in the QCD kernels and their integral contributions. Namely these terms contribute much to hard processes because they favour larger values of x. The stronger is their influence, the closer to zero should be the intercept of H_q with the abscissa axis. It would be interesting to get experimental information about the behaviour of H_q for soft and hard jets separately.

In conclusion, the separation of soft and hard jets according to the share of energy devoted to the "newly born" jet is proposed. If this is done, the experimentally measured values of mean multiplicities and other multiplicity distribution parameters of particles belonging to the soft jet can be compared with the obtained above theoretical predictions at different values of this share of energy. For a constant share, this dependence is the same as for the average total multiplicity but with non-trivial x_0-dependence of the factor in front of it. Some predictions are obtained for energy dependent cut-offs. The conclusions can be confronted to experiment.

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