Pis'ma v ZhETF, vol. 81, iss. 8, pp. 445 – 448 April 25

Simplicial vs. continuum string theory and loop equations

E. T. Akhmedov[1)]

Iinstitute of Theoretical and Experimental Physics, 117218 Moscow, Russia

Max–Planck–Institut für Gravitationsphysik, Albert–Einstein–Institut, 14476 Golm, Germany

Submitted 2 February 2005

We derive loop equations in a scalar matrix field theory. We discuss their solutions in terms of simplicial string theory – the theory describing embeddings of two-dimensional simplicial complexes into the space-time of the matrix field theory. This relation between the loop equations and the simplicial string theory gives further arguments that favor one of the statements of the paper hep-th/0407018. The statement is that there is an equivalence between the partition function of the simplicial string theory and the functional integral in a continuum string theory – the theory describing embeddings of smooth two-dimensional world-sheets into the space-time of the matrix field theory in question.

PACS: 11.10.–z, 11.15.Bt, 11.25.Tq

1. In this short note we give further arguments supporting the observations made in [1]. There we consider matrix scalar field theory in the D–dimensional Euclidian space:

$$Z=\int D\hat{\Phi}(x)D\hat{\bar{\Phi}}(x)\exp\left\{-\int d^Dx N\mathrm{Tr}\left[\frac{1}{2}\left|\partial_\mu\hat{\Phi}\right|^2+\right.\right.$$
$$\left.\left.+\frac{m^2}{2}\left|\hat{\Phi}\right|^2+\frac{\lambda}{3}\hat{\Phi}^3+\mathrm{c.c.}\right]\right\}, \qquad (1)$$

where $\mu=1,\ldots,D$, $\hat{\Phi}$ is $N\times N$ matrix field in the adjoint representation of $U(N)$ group: Φ^{ij}, $i,j=1,\ldots,N$. We choose this theory due to its simplicity (for our purposes) in comparison with gauge and matrix theories with more involved potentials. The problems of this theory due to the sign indefiniteness of its potential are irrelevant for our considerations: We consider this functional integral as a formal series expansion over λ. All our considerations can be easily generalized to the other matrix scalar and gauge theories. In fact, one can always make a theory with cubic interactions out of a theory with more involved interactions via insertions of integrations over additional fields into the functional integral.

The functional integral (1) can be transformed into the summation over the *closed* two-dimensional simplicial complexes[2)] [1]:

$$\log Z=\sum_{g=0}^{\infty}N^{\chi(g)}\sum_{V=0}^{\infty}\lambda^V\times$$
$$\times\sum_{\mathrm{graph;V,g\,fixed}}C'_{\mathrm{graph}}(V,g)\times$$
$$\times\left|\int_0^{+\infty}\prod_{n=1}^{L}\frac{d\alpha_n}{\alpha_n^{g\,D}}\,e^{-\frac{\alpha_n m^2}{2}}\int\prod_{a=1}^{F}d^D\mathbf{x}_a\times\right.$$
$$\left.\times\exp\left\{-\sum_{l=1}^{L}\frac{\alpha_l}{2}\left(\Delta_l\mathbf{x}\right)^2\right\}\right|_{\mathrm{graph}}, \qquad (2)$$

where $C'_{\mathrm{graph}}(V,g)$ are some combinatoric constants defined in [1]; F is the number of faces of the *fat* Feynman graph; L is the number of links; V is the number of vertices and g is the genus of the Feynman diagram.

The summation in eq.(2) is taken over the graphs which are dual to the Feynman diagrams [1]. These graphs represent triangulations of Riemann surfaces. In [1] we interpret the expression (2) as the partition function of the closed simplicial string theory – the theory describing embeddings of two-dimensional simplicial complexes into the space-time of the matrix field theory. In the context α's are related to the components of the two-dimensional metric [1].

Furthermore, in [1] we argue that there is no need to take a continuum limit in eq.(2): There should be a continuum string theory[3)] whose functional integral is equal to eq.(2). In this paper we give further arguments supporting this idea. We propose equations which are solved via the simplicial *open* string theory "functional

[1)] e-mail: akhmedov@itep.ru

[2)] Similar transformation has been done in [2] to establish a relation between the no-gravity limit of the Ponzano–Regge theory and a non-commutative field theory.

[3)] I.e. the theory describing embeddings of the smooth two-dimensional world-sheets into the space-time of the matrix field theory in question.

integral". On the other hand these equations have a natural interpretation as constraint equations in a two-dimensional field theory containing gravity.

To present the idea of our argument, let us consider the case of the relativistic particle. The path integral for the latter solves the following equation [3]:

$$\left(-\Delta + m^2\right)\, G(\mathbf{x},\, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}'). \tag{3}$$

One can also obtain a simplicial integral solution to this equation [1] as follows. The solution of eq.(3) can be represented as:

$$\begin{aligned} G(\mathbf{x},\, \mathbf{x}') &= \int d^D\mathbf{p}\; e^{\mathrm{i}\,\mathbf{p}\,(\mathbf{x}-\mathbf{x}')}\, \frac{1}{\mathbf{p}^2 + m^2} = \\ &= \frac{1}{\Lambda} \int d^D\mathbf{p}\; e^{\mathrm{i}\,\mathbf{p}\,(\mathbf{x}-\mathbf{x}')}\, e^{-\log\frac{\mathbf{p}^2+m^2}{\Lambda}} = \\ &= \frac{1}{\Lambda} \sum_{L=0}^{\infty} \frac{(-1)^L}{L!} \int d^D\mathbf{p} e^{\mathrm{i}\mathbf{p}(\mathbf{x}-\mathbf{x}')} \left[\log\frac{\mathbf{p}^2+m^2}{\Lambda}\right]^L = \\ &= \frac{1}{\Lambda} \sum_{L=0}^{\infty} \frac{(-1)^L}{L!} \int d^D\mathbf{p}\; e^{\mathrm{i}\,\mathbf{p}\,(\mathbf{x}-\mathbf{x}')} \times \\ &\times \prod_{l=1}^{L} \int_0^{+\infty} \frac{de_l}{e_l} \left(e^{-\frac{\mathbf{p}^2+m^2}{2}\, e_l} - e^{-\frac{\Lambda}{2}\, e_l}\right), \end{aligned} \tag{4}$$

where Λ is the cutoff. If we drop all terms containing $\exp\{-\Lambda\, e/2\}$ in eq.(4), we obtain the divergent expression which can be represented in the form [1]:

$$\begin{aligned} G_{\mathrm{div}}(\mathbf{x},\, \mathbf{x}') \propto \sum_{L=0}^{\infty} \frac{(-1)^L\, C_L}{L!} \int_0^{+\infty} \prod_{n=1}^{L} \frac{de_n}{e_n^{D/2+1}} \times \\ \times \int \prod_{i=1}^{L} d^D\mathbf{y}_i\, \exp\left\{-\frac{1}{2} \sum_{l=0}^{L} \left[\frac{(\Delta_l \mathbf{y})^2}{e_l} + m^2\, e_l\right]\right\}. \end{aligned} \tag{5}$$

In each member of the sum here $\mathbf{y}_0 = \mathbf{x}$, $\mathbf{y}_{L+1} = \mathbf{x}'$ and C_L are easily computable constants dependent on L.

The formula (5) contains the summation over the one-dimensional geometries. In fact, it contains the summation over all discretizations/triangulations (L) of the world-trajectory and the integration over all *one-dimensional* distances (e's) between the vertices (y's). The summation over the embeddings of the simplicial complexes is presented by the summation over the number of vertices (L) and the integration over all their possible positions, i.e. over – y's.

Thus, eq.(5) is, so to say, a simplicial particle theory "path integral" which formally solves eq.(3), but demands a regularization. At the same time eq.(4) suggests a natural regularization of the simplicial partition function (5) and rigorously relates it to the differential eq.(3). Moreover, as we see one does not have to take a continuum limit in eq.(5) and this "simplicial path integral" after the regularization is equivalent to the regularized standard path integral for the relativistic particle.

Below we argue that the same story should happen in the case of simplicial string theory and, possibly, for the higher dimensional simplicial brane theories. To obtain a proper theory of the latter kind one should both sum over the (multi-dimensional) triangulations and integrate over the sizes of the links: this gives the summation over all internal geometries, which in usual functional integrals is represented by the integration over all metrics divided by the volume of the group of diffeomorphisms.

2. Once the relation between eq.(1) and eq.(2) is established, one of the natural generalizations of eq.(3) to two-dimensions can be represented by the loop equations [4] in the matrix field theory. In this section we derive the loop equations for the theory (1) and discuss their obvious solution in terms of the simplicial open string theory. Such a string theory follows from the expansion in Feynman diagrams of an analog of the Wilson's loop correlation function [1]. As we will see, these loop equations have a natural interpretation as constrained equations on the functional integral for a continuum string theory. Obviously the latter should be equivalent to the simplicial string theory partition function in the same way as it happens in the case of the relativistic particle.

Thus, we would like to consider Ward type identities for the correlation function of the Wilson loop operator:

$$W(C) = \mathrm{TrP} \exp\left\{-\oint_C ds\, \sqrt{\dot{\mathbf{x}}^2(s)}\, \hat{\Phi}\left[x(s)\right]\right\}, \tag{6}$$

where C is a loop in the space-time, which is represented by the map $x(s)$. However, one can obtain closed[4)] loop equations for such an operator only in the theory with the Lagrangian [5]:

$$L = \frac{1}{2}\, \mathrm{Tr} \left|\partial_\mu \hat{\Phi}\right|^2 \tag{7}$$

or with the Lagrangians following from the reduction of the Yang–Mills theory. To obtain closed loop equations for the theory (1) we suggest to consider the loop operator as follows:

$$W(C, e) = \mathrm{TrP} \exp\left\{-\oint_C ds\, e(s)\, \hat{\Phi}\left[x(s)\right]\right\}. \tag{8}$$

As well there is the operator $\bar{W}(C, e)$ which depends on $\bar{\Phi}$ and the same e – real-valued square root of the one-dimensional internal metric on the interval of s.

[4)] Means equations which include no other kinds of operators except the loop ones.

Let us define the loop space Laplace operator as in [3]:

$$\frac{\partial^2}{\partial x^2(s)} = \int\limits_{s-0}^{s+0} ds' \frac{\delta^2}{\delta x_\mu(s)\,\delta x_\mu(s')}. \tag{9}$$

Then it is straightforward to see that [3, 4]:

$$\left(-\frac{\partial^2}{\partial x^2(s)} + m^2\, e(s)\,\frac{\partial}{\partial e(s)}\right) W(C,e) + \\ + \lambda\, e(s)\,\frac{\partial^2}{\partial e^2(s)} \bar{W}(C,e) = \\ = e(s)\,\mathrm{TrP}\left\{\left(-\partial_\mu^2 \hat{\Phi} + m^2\,\hat{\Phi} + \lambda\,\hat{\bar{\Phi}}^2\right) \times \right. \\ \left. \times \exp\left\{-\oint_C ds\, e(s)\,\hat{\Phi}\,[x(s)]\right\}\right\}. \tag{10}$$

Similarly one has the complex conjugate equation. To find the right hand side of this expression (after the averaging over all field configurations), let us consider the equality[5]:

$$0 = \int D\hat{\Phi}(x)\, D\hat{\bar{\Phi}}(x)\,\frac{\delta}{\delta\bar{\Phi}^a} \times \\ \times \left(\exp\left\{-\int d^D x\, N\,\mathrm{Tr}\left[\frac{1}{2}\left|\partial_\mu \hat{\Phi}\right|^2 + \right.\right.\right. \\ \left.\left.\left. \times \frac{m^2}{2}\left|\hat{\Phi}\right|^2 + \frac{\lambda}{3}\hat{\Phi}^3 + \mathrm{c.c.}\right]\right\} \times \right. \\ \left. \times \mathrm{TrP}\exp\left\{-\oint_C ds\, e(s)\,\hat{\bar{\Phi}}\,[x(s)]\right\}\right). \tag{11}$$

From this we obtain:

$$\left\langle\left(-\partial_y^2 \Phi^a(y) + m^2\,\Phi^a(y) + \lambda\,\left[\bar{\Phi}^2\right]^a(y)\right) \times \right. \\ \left. \times \exp\left\{-\oint_C ds\, e(s)\,\hat{\Phi}\,[x(s)]\right\}\right\rangle = \\ = -\left\langle \oint ds\, e(s)\,\delta\,[y - x(s)]\ \mathrm{P} \times \right. \\ \times \exp\left\{-\int\limits_x^y dt\, e(t)\,\hat{\bar{\Phi}}\,[x(t)]\right\} T^a \times \\ \left. \times \exp\left\{-\int\limits_y^x dt\, e(t)\,\hat{\bar{\Phi}}\,[x(t)]\right\}\right\rangle. \tag{12}$$

Here the left hand side appears from the variation over $\bar{\Phi}^a$ of the exponent of the action and the right hand side appears from the variation of $\bar{W}(C,e)$.

Hence, we obtain:

$$\left(-\frac{1}{e(s)}\,\frac{\partial^2}{\partial x^2(s)} + m^2\,\frac{\partial}{\partial e(s)}\right)\langle W(C,e)\rangle + \\ + \lambda\,\frac{\partial^2}{\partial e^2(s)}\left\langle \bar{W}(C,e)\right\rangle = \\ = \oint ds'\, e(s')\,\delta\,[x(s) - x(s')] \times \\ \times \left\langle \bar{W}(C_{xx'},\, e)\,\bar{W}(C_{x'x},\, e)\right\rangle \tag{13}$$

and the complex conjugate equation. In eq.(13) we use:

$$\sum_a T^a_{ij}\, T^a_{mn} = \delta_{in}\,\delta_{jm} \tag{14}$$

and $C^{xx}_{x'x'} = C_{x'x} \cup C_{xx'}$. The right hand side of eq.(13) does not vanish if the contour $C^{xx}_{x'x'}$ (which is just C with two designated points $x = x(s)$ and $x' = x(s')$) has self–intersection at $x = x'$ [3, 4].

The solution of eq.(13) via the expansion in powers of λ of the correlation function $\langle W(C,e)\rangle$ looks as follows:

$$\log\langle W(C,e)\rangle = \\ = \sum_{E=2}^{\infty} \int\limits_0^{2\pi} ds_1 e(s_1) \cdots \int\limits_0^{s_{E-1}} ds_E e(s_E) \times \\ \times \sum_{g=0}^{\infty} N^{\chi(g)} \sum_{V=0}^{\infty} \lambda^V \sum_{\mathrm{graph;V,g,E\,fixed}} C_{\mathrm{graph}}(E,V,g) \times \\ \times \left| \int\limits_0^{+\infty} \prod_{n=1}^{L} d\alpha_n \int \prod_{i=1}^{V} d^D \mathbf{y}_i \int \prod_{m=1}^{L} d^D \mathbf{p}_m \times \right. \\ \left. \times \exp\left\{-\sum_{l=1}^{L}\left[\frac{\alpha_l\,\left(\mathbf{p}_l^2 + m^2\right)}{2} - \mathrm{i}\mathbf{p}_l\,(\Delta_l \mathbf{y})\right]\right\}\right|_{\mathrm{graph}}, \tag{15}$$

where $C_{\mathrm{graph}}(E,V,g)$ are some combinatoric constants and in the exponent on the right hand side among the y's there are $y(s_1), \ldots, y(s_E)$ over which the integration is not taken and they are sitting on the contour C. The first sum on the right hand side is taken over their number. The summation over "graph" in eq.(15) means the summation over the Feynman diagram contributions to the correlation function in question. Accordingly, V is the number of interaction vertices; L is the number of propagators; y's are positions of the vertices; p's are momenta running over the propagators; α's are Schwinger parameters and g is the genus of the fat Feynman diagram.

Performing the transformation of [1], we obtain:

[5] Here $\hat{\Phi} = \Phi^a\, T^a$ and T^a, $a = 1, \ldots, N^2$ are the generators of $U(N)$.

$$\log\langle W(C,e)\rangle = \sum_{E=2}^{\infty}\int_0^{2\pi} ds_1 e(s_1)\cdots\int_0^{s_{E-1}} ds_E e(s_E)\times$$

$$\times\sum_{g=0}^{\infty} N^{\chi(g)}\sum_{V=0}^{\infty}\lambda^V \sum_{\text{graph;V,g,E fixed}} C'_{\text{graph}}(E,V,g)\times$$

$$\times\left|\int_0^{+\infty}\prod_{n=1}^{L}\frac{d\alpha_n}{\alpha_n^{(2g+1)\,D/2}}\,e^{-\frac{m^2\alpha_n}{2}}\int\prod_{a=1}^{F} d^D\mathbf{x}_a\times\right.$$

$$\times\exp\left\{-\sum_{l=1}^{L}\frac{\alpha_l}{2}\,(\Delta_l\mathbf{x})^2-\sum_{f=1}^{E}\frac{\mathbf{y}^2(s_f)}{2}\times\right.$$

$$\times\sum_{s,s'=1}^{2g+1}\omega_f^{(s)}\frac{1}{\sum_{l=1}^{L}\alpha_l\omega_l^{(s)}\omega_l^{(s')}}\omega_f^{(s')}+$$

$$\left.\left.+\mathrm{i}\sum_{f=1}^{E}\Delta_f\mathbf{x}\mathbf{y}(s_f)\right\}\right|_{\text{graph}}. \qquad (16)$$

Here $\omega_l^{(s)}$, $s=1,...,2g+1$ are the values on the l-th link of the closed (but not exact) one-forms on the genus g simplicial complex with one boundary. These simplicial complexes are defined by the dual graphs to the Feynman diagrams: Now the sum in eq.16 is taken over these dual graphs rather than the Feynman diagrams themselves. C'_{graph} is different from C_{graph} by a factor of the determinant of some matrix [1].

The main difference between eq.(13) and eq.(3) is that the former one is the non-linear equation. But dropping the right hand side of eq.(13) (and puting the functional δ-function instead), we obtain the standard linear Wheeler–DeWitt equation in a two-dimensional gravity theory coupled to the matter fields (x). Both loop and Wheeler–DeWitt equations are not well defined due to their divergences [3]. As the result, the solution of such equations in terms of two-dimensional functional integral is not known.

Note that the UV divergences of the quantum field theory in eq.(1) acquire a clear interpretation in the simplicial string theory description (16). These divergences are just due to the boundaries in the space of all metrics, i.e. when some of the α's vanish, which corresponds to the situations in which some of the triangles in the dual graph to the Feynman diagram degenerate into links [1]. The natural regularization of eq.(16) is analogous to the one presented in eq.(4) for the case of particle. It is nothing but the regularization which follows from the insertion of the integration over the ghost Pauli-Villars fields into the functional integral of the matrix field theory. The addition of these fields sets an obvious regularization of the loop equations, but one needs a renormalized version of these equations rather than just their regularization [3]. This is the subject for another work (see [6] for the attempts of understanding this point).

3. We have considered nonstandard loop variables in the scalar matrix field theory. These loop variables depend on both loops in the target space and internal one-dimensional metrics and obey loop equations. The equations represent a non-linear generalization of the Wheeler–DeWitt equations in a two-dimensional gravity theory interacting with matter. There is an obvious solution to these equations in terms of the partition function of an open simplicial string theory. We argue that there should be a continuum string theory solution to the same equations which is exactly equivalent to the simplicial one. The only obstacle which can appear in formulating such a continuum string theory is that for generic values of λ it can happen that its functional integral will contain an integration measure for the metrics which does not follow from a local norm.

I would like to acknowledge valuable discussions with Yu. Makeenko, M. Zubkov, F. Gubarev, V. Shevchenko, N. Amburg, T. Pilling and V. Dolotin. I would like to thank H. Nicolai, S. Theisen, A. Kleinschmidt, M. Zamaklar and K. Peeters for the hospitality during my visit to the MPI, Golm. This work was done under the partial support of grants RFBR # 04-02-16880, INTAS # 03-51-5460 and the Grant from the President of Russian Federation # MK-2097.2004.2.

1. E. T. Akhmedov, JETP Lett. **80**, 218 (2004) [Pisma Zh. Eksp. Teor. Fiz. **80**, 247 (2004)]; [arXiv:hep-th/0407018].
2. L. Freidel and E. R. Livine, arXiv:hep-th/0502106.
3. A. M. Polyakov, *Gauge Fields and Strings*, Harwood Academic Publishers, 1987.
4. Y. Makeenko and A. A. Migdal, Nucl. Phys. **B188**, 269 (1981) [Sov. J. Nucl. Phys. **32**, 431 (1980)].
5. Y. M. Makeenko, Phys. Lett. **B212**, 221 (1988).
6. E. T. Akhmedov, Phys. Lett. **B442**, 152 (1998), arXiv:hep-th/9806217; E. T. Akhmedov, arXiv:hep-th/0202055.