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Kelvin–Helmholtz instability in anisotropic superfluids

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Motivated by recent theoretical and experimental interest in the subject, we derive the condition of interfacial Kelvin–Helmholtz instability for a system of two flowing superfluids (one sliding on the other). The tensor structure of superfluid densities in anisotropic superfluids – such as ^{3}He-A, and also ^{3}He-B under external magnetic field – is properly taken into account. The consequences relevant to experiments on the $A-B$ phase boundary in superfluid ^{3}He are discussed.

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1. Introduction. The Kelvin-Helmholtz (KH) instability of an interface separating two flowing fluids manifests itself in various everyday phenomena in Nature, such as wave generation by wind blowing on water surface and flapping of flags and sails. A direct experimental verification of the theoretical prediction for the instability criterion, originally studied by Lord Kelvin [1], has nevertheless proven difficult with classical fluids. This is because viscous effects, neglected by the theory, affect the instability in an essential manner. However, as proven by experimental studies on the interface between the A and B phases of superfluid ^{3}He [2], in superfluid systems (where viscosity does not play a role) a well-defined instability can be observed, and the original theoretical ideas tested in detail.

At the instability of the $A-B$ interface, when a situation with shear flow is set up by rotating the sample, a small amount of quantized vorticity is transferred from the A phase to the B phase. Therefore, as a controllable vortex-injection mechanism, the KH instability has recently proven itself a valuable tool in various experimental studies concerning superfluid turbulence and the dynamics of quantized vortices in general [3]. Also, the dispersion relation for surface waves (ripplons) excited on the $A-B$ interface has been shown to be closely related to relativistic dynamics in the Schwarzschild metric [4]; in this work, the idea of using superfluid ^{3}He as a laboratory model system for testing some aspects of black-hole physics has been raised (see also Ref. [5]). Furthermore, the superfluid KH instability has been discussed in connection with multicomponent Bose–Einstein condensates [6], phase-separated ^{3}He–^{4}He mixtures [7], and even as a possible source for pulsar glitches [8].

Motivated by these recent developments, we derive the condition for the superfluid KH instability. Since both ^{3}He-A and ^{3}He-B in applied magnetic fields are anisotropic, we take into account their anisotropic superfluid densities. Also, fluid layers of arbitrary thickness are considered. The resulting instability criterion involves a different combination of density-tensor elements than suggested previously [9].

2. KH instability – main features. The problem of the KH instability in classical hydrodynamics considers an interface separating two immiscible ideal fluids in relative motion (one fluid sliding on the other, both of infinite extent), i.e. a tangential discontinuity. Such shear flow becomes unstable if the velocity difference exceeds a critical value determined by (see e.g. Ref. [11] for derivation)

$$\frac{1}{2}\,\frac{\rho_1\rho_2}{\rho_1+\rho_2}|\mathbf{v}_1-\mathbf{v}_2|^2=\sqrt{\sigma F}, \qquad (1)$$

where ρ_1 and ρ_2 ($\mathbf{v}_1$ and $\mathbf{v}_2$) are the densities (velocities) of the two fluids, σ is the surface tension of the separating interface, and F is a force (per unit volume) due to an external field stabilizing the position of the interface. Usually, this force is provided by the gravitational field, $F=g(\rho_1-\rho_2)$. The wave vector corresponding to the first unstable mode which gets excited at the instability is

$$k_0=\sqrt{F/\sigma}. \qquad (2)$$

The KH instability in the context of superfluids has been analyzed theoretically by Volovik [9]. As pointed out in this work, the most crucial modification as compared with the classical KH instability is the breakdown of the Galilean invariance, originating from the two-fluid nature of superfluid hydrodynamics. A preferred reference frame is provided by the frame of the container, with respect to which the normal components of the two superfluids are stationary. In this frame, assuming that

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$\mathbf{v}_1 \parallel \mathbf{v}_2$ (which corresponds to the experimental situation in Ref. [2]) the instability criterion emerges as [9]

$$\frac{1}{2}\,\rho_1 v_1^2 + \frac{1}{2}\,\rho_2 v_2^2 = \sqrt{\sigma F}, \tag{3}$$

with the same wave vector k_0 as in the classical case, given by Eq. (2). The densities and velocities in Eq. (3) refer to those of the superfluid components. Note that here the instability can also appear when $\mathbf{v}_1 = \mathbf{v}_2$ (cf. a flapping flag in the wind, where the flagpole breaks the Galilean invariance [10]).

Additionally, in the particular case of an $A{-}B$ interface in superfluid ^{3}He, the position of the interface is stabilized by an external magnetic-field gradient, $F = \frac{1}{2}(\chi_A - \chi_B)|\nabla(H^2)|$, where χ_A and χ_B are the susceptibilities of the two superfluid phases ($\chi_A > \chi_B$). Therefore, a well-defined instability occurs even though the mass densities of the phases are equal to a high accuracy. The threshold determined by Eq. (3) is in remarkable agreement with experimental observations [2].

3. KH instability of anisotropic superfluids. We now proceed to the derivation of the condition for superfluid KH instability, allowing for mass anisotropy and finite thicknesses of the two liquid layers. As shown in Ref. [9], the instability criterion can be derived in various different ways. Here we follow perhaps the most transparent of them, which considers the free energy connected with a perturbation of the interface between the two liquids. We take the unperturbed superfluid velocities $\mathbf{v}_{1(2)} = v_{1(2)}\,\hat{\mathbf{x}}$ of the liquids (in the rest frame of the container and the normal fractions) to be parallel to each other, and the coordinate z to be along the interface normal, see Fig.1. Translational invariance in

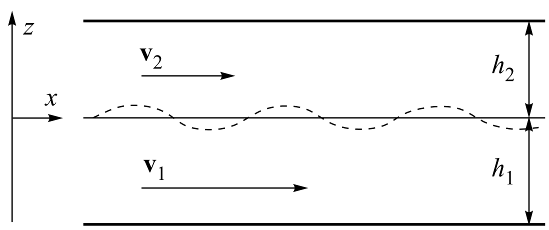

Fig.1. Geometry of the problem. Small-amplitude static perturbations of an interface at $z = 0$ separating two superfluid layers of thicknesses h_1, h_2 and superfluid velocities $\mathbf{v}_1$, $\mathbf{v}_2$ ($\mathbf{v}_1 \parallel \mathbf{v}_2$) are investigated

the y direction is assumed. We then write the superfluid velocities as $\mathbf{v}_{s1(2)} = \mathbf{v}_{1(2)} + \tilde{\mathbf{v}}_{1(2)}$, where $\tilde{\mathbf{v}}_{1(2)}$ are the modifications due to the perturbation of the interface.

The unperturbed interface is taken to be located at $z = 0$, and outer walls bounding the liquid layers situated at $z = -h_1$ and $z = h_2$. We consider small static interfacial perturbations uniform in the y direction, and of the form

$$\zeta = a\sin(kx). \tag{4}$$

The perturbation parts of the superfluid velocities can be written as $\tilde{\mathbf{v}}_{1(2)} = \nabla\psi_{1(2)}$, where

$$\begin{aligned}\psi_1 &= A_1\cosh[k_1^z(z+h_1)]\cos(kx),\\ \psi_2 &= A_2\cosh[k_2^z(z-h_2)]\cos(kx),\end{aligned} \tag{5}$$

which satisfies $\tilde{v}_{1,z}(z=-h_1) = \tilde{v}_{2,z}(z=h_2) = 0$ at the solid outer boundaries. From the equations of continuity

$$\nabla\cdot(\overline{\boldsymbol{\rho}}_{s1}\cdot\mathbf{v}_{s1}) = 0, \quad \nabla\cdot(\overline{\boldsymbol{\rho}}_{s2}\cdot\mathbf{v}_{s2}) = 0, \tag{6}$$

where $\overline{\boldsymbol{\rho}}_{s1(2)} = \rho^x_{1(2)}\hat{\mathbf{x}}\hat{\mathbf{x}} + \rho^y_{1(2)}\hat{\mathbf{y}}\hat{\mathbf{y}} + \rho^z_{1(2)}\hat{\mathbf{z}}\hat{\mathbf{z}}$ is the anisotropic superfluid density tensor, we obtain the conditions

$$\rho_1^x\;k^2 = \rho_1^z\;(k_1^z)^2, \quad \rho_2^x\;k^2 = \rho_2^z\;(k_2^z)^2. \tag{7}$$

Additionally, we require that there be no mass flow through the interface,

$$\hat{\mathbf{s}}\cdot(\overline{\boldsymbol{\rho}}_{s1}\cdot\mathbf{v}_{s1}) = 0, \quad \hat{\mathbf{s}}\cdot(\overline{\boldsymbol{\rho}}_{s2}\cdot\mathbf{v}_{s2}) = 0, \tag{8}$$

where $\hat{\mathbf{s}}$ is the unit normal of the interface, we find the further conditions (to first order in the small perturbation)

$$\begin{aligned}\rho_1^z A_1 k_1^z\sinh(k_1^z h_1) - \rho_1^x a v_1 k = 0,\\ \rho_2^z A_2 k_2^z\sinh(k_2^z h_2) + \rho_2^x a v_2 k = 0.\end{aligned} \tag{9}$$

We return to the justification of Eq. (8) in more detail below.

The free-energy functional for the perturbed flow can be written in the form

$$\mathcal{F}[\zeta] = \frac{1}{2}\int dx\,\left[F\zeta^2 + \sigma\left(\frac{d\zeta}{dx}\right)^2 + \int_{-h_1}^{\zeta} dz(\mathbf{v}_{s1}\cdot\overline{\boldsymbol{\rho}}_{s1}\cdot\mathbf{v}_{s1}) + \int_{\zeta}^{h_2} dz(\mathbf{v}_{s2}\cdot\overline{\boldsymbol{\rho}}_{s2}\cdot\mathbf{v}_{s2})\right].$$

The flow is unstable when the free energy of perturbed flow is lower than that of unperturbed flow, i.e. when $\mathcal{F}[\zeta] < \mathcal{F}_0 \equiv \mathcal{F}[\zeta = 0]$. Substituting Eqs. (4)–(9), we find that the first-order modification of the free energy vanishes, and the second-order contribution reads

$$\mathcal{F}[\zeta] - \mathcal{F}_0 \propto a^2\left[F + \sigma k^2 - k(\rho_1^{\text{eff}}v_1^2 + \rho_2^{\text{eff}}v_2^2)\right], \tag{10}$$

with the definition

$$\rho_1^{\text{eff}} \equiv \frac{\rho_1^x\sqrt{\rho_1^z/\rho_1^x}}{\tanh\left(kh_1\sqrt{\rho_1^x/\rho_1^z}\right)}, \tag{11}$$

and similarly for ρ_2^{eff}. The criterion for the instability is now determined so that the expression on the right-hand side of Eq. (10) first becomes negative for some wave vector k. In the limiting case of thick layers (appropriate for the experiments in Ref. [2]), $kh_1, kh_2 \gg 1$, this happens first for $k = k_0$, and the instability condition reads

$$\frac{1}{2}\left(\rho_1^x\sqrt{\frac{\rho_1^x}{\rho_1^z}}\ v_1^2 + \rho_2^x\sqrt{\frac{\rho_2^x}{\rho_2^z}}\ v_2^2\right) = \sqrt{\sigma F}. \qquad (12)$$

In the isotropic limit, $\rho_{1(2)}^x = \rho_{1(2)}^z = \rho_{1(2)}$, the criterion in Eq. (3) is recovered. The full anisotropic result, however, differs from Eq. (21) in Ref. [9].

Another limiting case which could be experimentally realized, as well as being interesting in view of Ref. [4], is that of one thin layer, say $kh_1 \ll 1$, and $v_2 = 0$. In this case, the parameters of fluid 2 (h_2, ρ_2^x and ρ_2^z) do not enter the instability criterion. It then follows that the instability first develops with large wavelengths, $k \to 0$, and the threshold velocity adopts the simple form

$$v_1 = \sqrt{Fh_1/\rho_1^x}. \qquad (13)$$

4. Boundary condition. To give a physical motivation to the boundary condition in Eq. (8), we return again to the specific case of the $A-B$ phase boundary in superfluid ^{3}He. Experimentally, a situation with shear flow can be accomplished by rotating a sample of superfluid ^{3}He where an $A-B$ interface has initially been stabilized using an external magnetic field with a gradient along the axis of rotation, see Fig.2. Because the critical velocity of vortex nucleation is much lower in ^{3}He-A than in ^{3}He-B, with moderate angular velocities of rotation vortex lines appear in the A-phase volume while the volume occupied by the B phase remains vortex-free. In this way, a relative flow between the superfluid components of these two quantum liquids is set up.

Since the vortices cannot terminate at the interface, they must bend to the container wall. Actually, the vortices form a surface sheet on the phase boundary [12]. In stable equilibrium, the net force on the vortex lines coating the interface must vanish when they are stationary in the container frame, *i.e.* $\mathbf{v}_L = \mathbf{v}_n$. In that case, there is no frictional force from the normal component, and the equation of force balance reads (see, e.g. Ref. [13])

$$\mathbf{F}_{\text{tot}} = \mathbf{F}_{\text{M}} + \mathbf{F}_{\text{I}} + \mathbf{F}_{\text{int}} = 0, \qquad (14)$$

with the Magnus force from the local superfluid velocity field

$$\mathbf{F}_{\text{M}} = \rho\boldsymbol{\kappa} \times (\mathbf{v}_L - \mathbf{v}_s), \qquad (15)$$

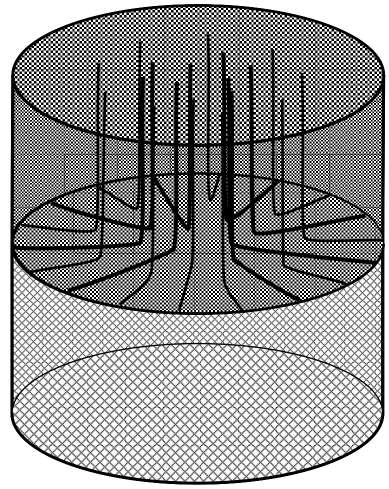

Fig.2. Schematic representation of the $A-B$ interface instability experiment in superfluid ^{3}He in a state of equilibrium. In a rotating cylindrical container, the A-phase occupying the upper volume contains the equilibrium number of vortices, and the lower B-phase volume remains vortex-free. The A-phase vortices bend to the container wall forming a vortex sheet on the interface

where ρ is the total mass density of the liquid and $\boldsymbol{\kappa}$ is the circulation vector; the Iordanskii force from the elementary excitations (quasiparticles in the system)

$$\mathbf{F}_{\text{I}} = \boldsymbol{\kappa} \times [\overline{\boldsymbol{\rho}}_n \cdot (\mathbf{v}_s - \mathbf{v}_n)], \qquad (16)$$

(with the normal-density tensor $\overline{\boldsymbol{\rho}}_n$) and the force from the interface

$$\mathbf{F}_{\text{int}} = f_{\text{int}}\ \hat{\mathbf{s}}. \qquad (17)$$

With these definitions, the vanishing of $\hat{\mathbf{s}} \times \mathbf{F}_{\text{tot}}$ implies Eq. (8). Obviously, for the condition of local equilibrium to be valid, in the above derivation we have assumed that the timescale characterizing the dynamics of vortices is short compared with that determining the time evolution of the $A-B$ interface.

5. Consequences. Next, we discuss the implications of Eq. (12) regarding the $A-B$ phase boundary experiments of the type reported in Ref. [2]. Since the A phase essentially contains the equilibrium number of vortices, $v_A \approx 0$, and the critical velocity of the KH instability is given by

$$v_B = \frac{(4\sigma_{AB}F)^{1/4}}{\sqrt{\rho_B^{\text{eff}}}}, \qquad (18)$$

where now $\rho_B^{\text{eff}} = \rho_B^x\sqrt{\rho_B^x/\rho_B^z}$. Despite the anisotropy inherent in the p-wave pairing of superfluid ^{3}He, unperturbed bulk B-phase is isotropic in its physical properties. In the presence of an external magnetic field and superflow, however, gap distortion induces an anisotropy to the superfluid density in ^{3}He-B [14]. The resulting density tensor is of uniaxial form, with components given by

$$\rho_{B,ij} = \rho_B^{\|}\ \hat{l}_i\hat{l}_j + \rho_B^{\perp}\ (\delta_{ij} - \hat{l}_i\hat{l}_j), \tag{19}$$

where the unit vector $\hat{\mathbf{l}} \equiv \hat{\mathbf{H}} \cdot \overline{\mathbf{R}}$ is the axis of orbital anisotropy ($\hat{\mathbf{H}}$ is a unit vector in the direction of the external magnetic field and $\overline{\mathbf{R}}$ the rotation matrix defining the B-phase order parameter).

In the Ginzburg–Landau regime, the mass-density tensor components in the presence of an external field H can be written as [14]

$$\rho_B^{\|} \approx \left[1 - 3\frac{H^2}{H_0^2(1 - T/T_c)}\right]\rho_B^0, \qquad \rho_B^{\perp} \approx \rho_B^0, \tag{20}$$

where $H_0 = p_F/m^*\xi_0\gamma \approx 1.64$ T, and ρ_B^0 is the isotropic value corresponding to unperturbed bulk ^{3}He-B. In first approximation, therefore, the tensor component along the direction of $\hat{\mathbf{l}}$ is suppressed, while the other components stay unaffected. We have neglected the small additional suppression due to counterflow, which is justified for the typical experimental velocities $v_B \ll \ll (2m^*\xi_0)^{-1} \approx 6.3$ cm/s.

To estimate the magnitude of the effect, we insert the values $H = 367$ mT and $T = 0.57\ T_c$ (one particular experiment performed at pressure $p = 29$ bar, see Ref. [3], reference 30) in Eq. (20), leading to $\rho_B^{\|} \approx 0.65\ \rho_B^0$. Although the extrapolation of the Ginzburg–Landau result to such low temperatures can certainly be questioned, we think it is safe to conclude that the anisotropy effects discussed here are large enough to have experimental significance.

Of course, the actual effect of the density anisotropy depends on the orientation of $\hat{\mathbf{l}}$ in our coordinate system depicted in Fig.1. This requires a careful analysis of several different mechanisms trying to orient the order parameter [15], originating e.g. from the external magnetic field, counterflow, container surfaces (the instability is expected to occur near the surface where the counterflow is highest), and the presence of the $A-B$ interface. Because a detailed investigation of these effects is a fairly complicated problem, we list three possible orientations of $\hat{\mathbf{l}}$, which correspond to the preferred directions of different orienting influences.

(i) $\hat{\mathbf{l}} \parallel \hat{\mathbf{x}}$: This choice minimizes the kinetic energy of the flow; the axis of orbital anisotropy coincides with the flow direction, and we have

$$\rho_B^{\text{eff}}(\hat{\mathbf{l}} \parallel \hat{\mathbf{x}}) = \rho_B^{\|}\sqrt{\rho_B^{\|}/\rho_B^{\perp}}. \tag{21}$$

According to Eq. (20), this results in a reduction of ρ_B^{eff} as compared to the isotropic value ρ_B^0, and an enhancement of the threshold velocity in Eq. (18).

(ii) $\hat{\mathbf{l}} \parallel \hat{\mathbf{y}}$: For this wall-dominated order-parameter orientation,

$$\rho_B^{\text{eff}}(\hat{\mathbf{l}} \parallel \hat{\mathbf{y}}) = \rho_B^{\perp}, \tag{22}$$

and the main gap suppression is along the direction perpendicular to the plane of Fig.1. In this case, therefore, no significant deviation from the isotropic result is to be expected.

(iii) $\hat{\mathbf{l}} \parallel \hat{\mathbf{z}}$: This orientation would follow in the absence of other effects than that of the axially oriented magnetic field. We obtain

$$\rho_B^{\text{eff}}(\hat{\mathbf{l}} \parallel \hat{\mathbf{z}}) = \rho_B^{\perp}\sqrt{\rho_B^{\perp}/\rho_B^{\|}}, \tag{23}$$

resulting in an apparent *enhancement* of the effective superfluid density from the isotropic value, $\rho_B^{\text{eff}} > \rho_B^0$! With the values of H and T used in the earlier estimate above, we find $\rho_B^{\text{eff}} \approx 1.25\ \rho_B^0$. It is interesting to note that the authors of Ref. [3] state (reference 30 in the article) that a good fit to the experimental data of the instability threshold, using Eq. (3), was obtained by taking $\rho_B(H) \approx 1.15\ \rho_B(H = 0)$.

Even though the choice $\hat{\mathbf{l}} \parallel \hat{\mathbf{z}}$ appears difficult to justify in the circumstances of the experiment (the $A-B$ phase boundary has a strong tendency to orient $\hat{\mathbf{l}} \perp \hat{\mathbf{s}}$ [16]), it is clear that any attempts aiming at a quantitative understanding of the $A-B$ interface instability should take anisotropy effects into account.

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