

# Comment on the surface exponential for tensor fields

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Starting from essentially commutative exponential map  $E(B|I)$  for generic tensor-valued 2-forms  $B$ , introduced in [10] as direct generalization of the ordinary non-commutative  $P$ -exponent for 1-forms with values in matrices (i.e. in tensors of rank 2), we suggest a non-trivial but multi-parametric exponential  $\mathcal{E}(B|I|t_\gamma)$ , which can serve as an interesting multi-directional evolution operator in the case of higher ranks. To emphasize the most important aspects of the story, construction is restricted to backgrounds  $I_{ijk}$ , associated with the structure constants of commutative associative algebras, what makes it insensitive to topology of the  $2d$  surface. Boundary effects are also eliminated (straightforward generalization is needed to incorporate them).

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**1. Motivations.** Given a matrix-valued one-form  $A_{ij}^\mu(x)dx_\mu$  on a line, one can introduce an ordered exponent,  $P \exp(\int A_{ij}(x)dx)$ , which can be also defined as a limit

$$\begin{aligned} z(\hat{T} + \hat{A}) &\equiv P \exp\left(\int A_{ij}(x)dx\right) = \\ &= \lim_{N \rightarrow \infty} \prod_{n=1}^N \left(\hat{T} + \frac{1}{N} \hat{A}(n/N)\right), \end{aligned} \quad (1)$$

with the unit matrix  $I_{ij} = \delta_{ij}$  (hats denote tensors with suppressed indices).

When a  $1d$  line (real curve) is substituted by  $2d$  surface (complex curve)  $\Sigma$ , appropriate generalization of  $P$ -exponent is badly needed for numerous string theory considerations. The problem is known under many names, from topological models [1, 2] to Connes-Kreimer theory [3–5] and that of the 2-categories [6]. The role of a chain in (1) is now played by a “net” – a dual graph  $\Gamma$ , describing simplicial complex, which “triangulates”<sup>1)</sup>  $\Sigma$ , and tensors  $\hat{T}^{(m)}$  of rank  $m$  stand as coupling constants at the vertices of valence  $m$  of this graph. To provide

triangulation of a surface, the graph should have many vertices with valencies  $m \geq 3$ , and essential generalization of (1) is unavoidable. Indices of  $\hat{T}$ 's at the vertices are contracted by special rank 2 tensors  $\hat{g}$ , called metric or propagator (it is assumed to have upper indices if all  $\hat{T}$  have lower ones). With such a graph one associates a partition function [4]  $\hat{Z}_\Gamma^{\hat{g}}\{\hat{T}\}$  which is a tensor of rank  $\text{ext}(\Gamma)$ , equal to the number of external legs of the graph  $\Gamma$  (if  $\Sigma$  has no boundaries,  $\text{ext}(\Gamma) = 0$ ).

**2. Suggestion of ref.[10].** In [10] a principally important step is made to bring the abstract constructions of [4] and [11–13] closer to appropriate generalization of (1). The crucial additional structure, used in (1), is decomposition of the rank 2 tensor  $\hat{t} = \hat{T} + \hat{A}$  into “background”  $\hat{T}$  and “dynamical”  $\hat{A}$  parts.  $z(\hat{T}) = 1$  is trivial, though  $I_{ij} = \delta_{ij} \neq 0$  itself is not.

For complex curves the background  $\hat{T}$  should necessarily have rank greater than 2, for graphs of valence 3 it should have rank 3. It is well known [2], that there are non-trivial  $\hat{T}$  and associated  $\hat{g}$  exist with trivial

$$Z_\Gamma^{\hat{g}}(\hat{T}) = 1$$

for all  $\Gamma$  without external legs<sup>2)</sup> (they can be build from structure constants of any commutative associative algebra, see s. below, but this does not exhaust all the possibilities), and ref.[10] suggests to make use of them exactly in the same way as in (1). Since in this construction  $\hat{g}$  is rigidly linked to  $\hat{T}$ , we suppress the  $\hat{g}$  labels in most formulas below.

<sup>1)</sup>In principle, to imitate surface integrals with arbitrary measures one needs triangulations with lengths, ascribed to the links [7]. However, as usual in the matrix-model realizations of string theory [8], one can ignore the lengths (consider “equilateral triangulations”) – like it is done in (1) – and associated quantities will be quite informative (perhaps, even *exhaustively* informative: for example, it is believed – and confirmed by numerous calculations of particular quantities – that Polyakov's (sum over  $2d$  metrics) and Migdal's (matrix-model sum over equilateral triangulations) descriptions of string correlators are equivalent). Still, the interrelation between arbitrary and equilateral triangulations remains an open problem, touching the fundamental questions of number theory [9]. The problem is also known as that of continuous limits in matrix models.

<sup>2)</sup>Of course, for  $\text{ext}(\Gamma) \neq 0$  the  $\hat{Z}_\Gamma^{\hat{g}}(\hat{T})$  is an operator (has external indices) and can not be unity. However, it can be made dependent only on  $\text{ext}(\Gamma)$  but not on  $\Gamma$  itself, see s. below.

According to the definition of  $\widehat{Z}_\Gamma$  for  $\widehat{T} = \widehat{I} + \widehat{B}$  we have<sup>3)</sup>:

$$\widehat{Z}_\Gamma(\widehat{I} + \widehat{B}) = \sum_{\gamma \subset \Gamma} \widehat{Z}_{\Gamma/\gamma}(\widehat{I}) \widehat{Z}_\gamma(\widehat{B}). \quad (2)$$

If we now take a limit of large  $|\Gamma| \equiv (\# \text{ of vertices in } \Gamma)$ , with graph growing in both dimensions to form a dense net, and look at the terms with a given power of  $B$ , then statistically only graphs  $\gamma$  consisting of isolated points will survive after appropriate rescaling of  $B$ , and this logic leads to the following generalization of (1) [10]:

$$\widehat{E}(\widehat{B}|\widehat{I}) = \lim_{|\Gamma| \rightarrow \infty} \widehat{Z}_\Gamma \left( \widehat{I} + \frac{1}{|\Gamma|} \widehat{B} \right) \quad (3)$$

(the argument  $\widehat{I}$  in  $\widehat{E}$  will often be suppressed below). This is a very nice and interesting quantity, but it is essentially Abelian: as we shall see in (11),  $\widehat{E}(\widehat{B}_1)\widehat{E}(\widehat{B}_2) = \widehat{E}(\widehat{B}_1 + \widehat{B}_2)$  (for example, for a rank 3 2-form  $\widehat{B} = B_{ij}^{\mu\nu} dx_\mu \wedge dx_\nu$  on  $\Sigma$  we can define a surface integral as  $E \left( \int_\Sigma \widehat{B} \right)$  and never encounter any ordering problems). This happens for the same reason that the homotopic groups  $\pi_k$  are commutative for  $k > 1$ : any two insertions of  $B$  at two *remote* points can be easily permuted, moving one *around* another.

In what follows we give a more formal description of above construction, getting rid of a subtle limiting procedure in (3), and introduce – with the help of  $\widehat{E}(\widehat{B})$ , just changing its argument – a less trivial exponential  $\widehat{E}(\widehat{B}|t)$ . It should be useful in applications, it is well defined, but no “conceptual” limiting formula like (3) is immediately available for it.

**3. Backgrounds  $\widehat{I}$  from commutative associative algebras  $\mathcal{A}$ .** Let  $(\check{C}_i)_j^k = C_{ij}^k$  be structure constants of an associative algebra  $\mathcal{A}$  ( $\phi_i * \phi_j = C_{ij}^k \phi_k$ ):

$$[\check{C}_i, \check{C}_j] = 0. \quad (4)$$

Introduce a set of *symmetric* tensors  $\widehat{I}^{(n)}$ :

<sup>3)</sup>Note that the graph automatically picks up the tensors of appropriate rank from  $\widehat{B}$ , if there is no match,  $\widehat{Z} = 0$ . If we assume that  $\widehat{I}$  is exactly of rank 3 while  $\widehat{B}$  consists of tensors of various ranks (we’ll see below that it is useful not to restrict  $\widehat{B}$  to rank 3 only), then only subgraphs  $\Gamma/\gamma$  with vertices of valence 3 will contribute to the sum. The subgraph  $\Gamma/\gamma$  is defined by throwing away all the vertices of  $\gamma$  and all the links between them,  $\text{ext}(\Gamma/\gamma) = \text{ext}(\Gamma) + \text{ext}(\gamma) - 2(\# \text{ of common external legs of } \Gamma \text{ and } \gamma)$ . In other words,  $\gamma$  is treated as “vertex-subgraph” of  $\Gamma$ . As explained in [4], the “vertex-subgraphs” (in variance with the “box-subgraphs”) are related to relatively simple set-theoretical aspect of quantum field theory (to the *Shift*  $\mathcal{M}$  rather than *Diff*  $\mathcal{M}$  structure of diagram technique). In the present context this is the reason behind the over-simplicity (commutativity property (11)) of the exponential (3).

$$I_{i_1 \dots i_n} \equiv \text{Tr}(\check{C}_{i_1} \dots \check{C}_{i_n}). \quad (5)$$

Among them will be the **metric**<sup>4)</sup>

$$g_{ij} = I_{ij} = \text{Tr}(\check{C}_i \check{C}_j) \quad (6)$$

and the **elementary vertex**  $I_{ijk} = \text{Tr}(\check{C}_i \check{C}_j \check{C}_k)$ . Metric will be assumed non-degenerate and its inverse  $g^{ij}$  will be used to raise indices.

In what follows we impose additional **commutativity** condition on the structure constants:

$$C_{ij}^k = C_{ji}^k. \quad (7)$$

Then tensors  $\widehat{I}$  are not just cyclic, but totally symmetric.

**Lemma:** For commutative associative algebra  $\mathcal{A}$

$$I_{ijk} = g_{im} C_{jk}^m$$

or simply  $I_{jk}^m = C_{jk}^m$ . Indeed,

$$g_{im} C_{jk}^m = C_{il}^n C_{mn}^l C_{jk}^m \stackrel{(7)}{=} C_{il}^n C_{nm}^l C_{jk}^m \stackrel{(4)}{=}$$

$$\stackrel{(4)}{=} C_{il}^n C_{jm}^l C_{nk}^m = I_{ikj} \stackrel{(7)}{=} I_{ijk}.$$

**Lemma:** For commutative associative algebra  $\mathcal{A}$

$$I_{i_1 \dots i_m k_1 \dots k_r} I_{j_1 \dots j_n}^{k_1 \dots k_r} =$$

$$I_{i_1 \dots i_m k_1 \dots k_r} I_{j_1 \dots j_n \tilde{k}_1 \dots \tilde{k}_r} g^{k_1 \tilde{k}_1} \dots g^{k_r \tilde{k}_r} = I_{i_1 \dots i_m j_1 \dots j_n} \quad (8)$$

for any  $r \neq 0$  and any  $m$  and  $n$ . The proof follows from the observation that (4) and (6) provide the two transformations (the “flip” or “zigzag” transform and tadpole-eliminator), which generate a group with transitive action on the space of all connected triangulations (see [10] for the relevant illustrations).

Obviously, for any *connected*  $\Gamma$ ,  $\widehat{Z}_\Gamma(\widehat{I})$  is given by these tensors  $I$ :

$$\left( Z_\Gamma^\delta(\widehat{I}) \right)_{i_1 \dots i_{\text{ext}(\Gamma)}} = I_{i_1 \dots i_{\text{ext}(\Gamma)}}. \quad (9)$$

In other words, in background theory the connected diagram depends only on the number of external legs. One can of course associate additional factors with graphs, counting the numbers of vertices and loops, but they do not depend on  $\mathcal{A}$  and are not of immediate interest for our consideration.

If  $\Gamma$  consists of disconnected parts,  $\widehat{Z}_\Gamma(\widehat{I})$  will be a tensor product of  $\widehat{I}$ -tensors.

If other *traceless* tensors  $B^{(m)}$  of rank  $m$  are allowed in the vertices, we get a non-trivial  $B$ -theory in the  $\mathcal{A}$ -background. Original  $\widehat{Z}(\widehat{T})$  provides the unified background-independent formulation. Still, explicit

<sup>4)</sup>Note that this choice of metric is *different* from  $G_{ij}^{(m)} = I_{ij}^{(3)}$ , used in the context of the generalized WDVV equations in [14].

transformation from one background to another remains an interesting open problem.

**4. Commutative exponential.** Introduce a  $\widehat{B}$ -dependent tensor of rank  $n$

$$E_{i_1 \dots i_n}^{(n)}(\widehat{B}) = \sum_{m=1}^{\infty} \sum_{\{r_m\}} \sigma\{r_m\} \times \left\{ \left( B^{j_1^{(1)}} \dots B^{j_{r_1}^{(1)}} \right) \left( B^{j_1^{(2)}} j_2^{(2)} \dots B^{j_{2r_2-1}^{(2)} j_{2r_2}^{(2)}} \right) \dots \left( B^{j_1^{(m)}} \dots j_m^{(m)} \dots B^{j_{r_m-m+1}^{(m)} \dots j_{r_m}^{(m)}} \right) \dots \right\} I_{i_1 \dots i_n j_1^{(1)} \dots j_{r_m}^{(m)} \dots} \quad (10)$$

Here  $\widehat{B} = \{\widehat{B}^{(1)}, \widehat{B}^{(2)}, \dots, \widehat{B}^{(m)}, \dots\}$  is a direct sum of tensors of all possible ranks and the sum in (11) is over all possible sets, including any number  $r_m$  of tensors of rank  $m$ .

**Theorem:** For appropriate choice of the combinatorial factors  $\sigma\{r_m\}$  the map  $\widehat{E}(\widehat{B})$  satisfies the exponential property<sup>5)</sup>:

$$E_{i_1 \dots i_m k_1 \dots k_r}^{(m+r)}(B_1) E_{j_1 \dots j_n}^{(n+r) k_1 \dots k_r}(B_2) = E_{i_1 \dots i_m j_1 \dots j_n}^{(m+n)}(B_1 + B_2) \quad (11)$$

for any  $r \neq 0$  and any  $m$  and  $n$  (no sum over  $r$  is taken). The relevant choice of  $\sigma\{r_m\}$  is the usual Feynman-diagram factorial (see, for example, the ‘‘generalized Wick theorem’’ in [15])

$$\sigma\{r_m\} = \prod_{m=1} \frac{1}{r_m! (m!)^{r_m}}. \quad (12)$$

The factors  $m!$  can be eliminated by rescaling of  $B^{(m)}$ . As immediate corollary of (11), derivative of the exponent functor  $\widehat{E}(\widehat{B})$  is  $\widehat{E}(\widehat{B})$  itself:

$$\delta E_{i_1 \dots i_n}^{(n)}(B) = \sum_{m=1}^{\infty} E_{i_1 \dots i_n j_1 \dots j_m}^{(n+m)}(B) \delta B_{(m)}^{j_1 \dots j_m} + O(\delta B^2) \quad (13)$$

Thus nothing like non-trivial Campbell–Hausdorff formula [16] (which describes the product of  $P$ -exponents (1)) arises for  $\widehat{E}(\widehat{B})$ , potential non-commutativity of tensor product is completely eliminated by the naive continuum limit (3), as a corollary of the relation (9) and the possibility to rely upon *connected* graphs.

**5. Non-trivial exponential and directions of immediate generalization.** Potential origin of non-triviality of the exponential is two-fold: there can be contributions from non-trivial (not totally disconnected)

<sup>5)</sup>Note that in terms of  $\widehat{T} = \widehat{I} + \widehat{B}$  this relation does not look homogeneous:

$$\widehat{Z}_\Gamma(\widehat{I} + \widehat{B}_1 + \widehat{B}_2) = \widehat{Z}_\Gamma(\widehat{I} + \widehat{B}_1) \widehat{Z}_\Gamma(\widehat{I} + \widehat{B}_2) + O(|\Gamma|^{-1}),$$

and holds because of the  $\Gamma$ -independence property (9) of  $\widehat{Z}_\Gamma(\widehat{I})$ .

subgraphs  $\gamma$  to  $\widehat{Z}_\gamma(\widehat{B})$  and from *disconnected* factor-graphs  $\Gamma/\gamma$  to  $\widehat{Z}_{\Gamma/\gamma}(\widehat{I})$ . The first origin (contribution from non-trivial  $\gamma$ ) is eliminated by naive continuum limit – both in  $1d$  formula (1) and in the  $2d$  one (11). In  $1d$  the property (9) perfectly holds for *connected* graphs, but disconnected  $\Gamma/\gamma$  also contribute to (1). What happens in  $2d$  is that disconnected  $\Gamma/\gamma$  are statistically damped in the naive continuum limit, *together* with non-trivial  $\gamma$  – and direct generalization of the non-Abelian (1) from lines to surfaces is Abelian (commutative)!

In order to get a non-commutative exponential in  $2d$  one can, however, revive the contributions from *non-trivial*  $\gamma$ , simply by introducing a non-trivial multi-time evolution operator:

$$\widehat{\mathcal{E}}(B|t) = \widehat{E} \left( \sum_{\text{connected } \gamma} t_\gamma \widehat{Z}_\gamma(B) \right) \quad (14)$$

(of course, one can do – and often does – the same in  $1d$ ). Similarly, one can add contributions from disconnected  $\Gamma/\gamma$  by introducing certain non-local operators (involving contour integrals) in the exponent. Despite such quantities may seem less natural than (3), they naturally arise in physically relevant evolution operators and even in actions, bare and effective. Moreover, for special  $\widehat{B}$ , for example, totally antisymmetric, the leading contribution with single-point  $\gamma$  vanishes in *symmetric* background  $\widehat{I}$ . Then the next-to-leading contribution – from single-link (and two-point)  $\gamma$  – can be described by appropriately modified limiting prescription (3).

$\widehat{\mathcal{E}}(\widehat{B}|\widehat{I}|t)$  is already a non-trivial (operator) special function, which deserves attention and investigation. Note, that even if  $\widehat{B}$  was a rank-3 tensor, and all the relevant graphs were of valence 3, the tensors  $\widehat{Z}_\gamma(\widehat{B})$  which contribute to the argument of  $\widehat{E}$  in (14) have ranks  $\text{ext}(\gamma)$ , not obligatory equal to 3.

Despite non-trivial graphs  $\gamma$  are now incorporated in (14), they are still restricted to *lie inside*  $\Sigma$ , remain separated from the boundary of  $\Sigma$  by requirement of connectedness of  $\Gamma/\gamma$  (external legs of  $\Gamma$  are not allowed to belong to  $\gamma$ ). Additional corrections to (14) are needed to make it sensitive to boundary effects.

In this note we restricted consideration to the simplest possible case of commutative algebra  $\mathcal{A}$ , when tensors  $\widehat{I}$  are totally symmetric. Relaxing this requirement, one gets  $\widehat{I}$  with only cyclic symmetry, then (9) gets more complicated: universality classes are no longer enumerated by  $\text{ext}(\Gamma)$ , dependence on the number of handles arises and description in terms of fat graphs is needed [10] (see [5] for the corresponding generalization of [4]).

An interesting part of this story is exponentiation (the algebra  $\rightarrow$  group lifting) of associative algebras and

higher-rank multiplications [12]. It involves limits like (3) along peculiar chains of graphs (obtained, for example, by iterative blowing up of triple vertices into triangles). In such situations, the  $B$  insertions have higher probability to break the graph into disconnected components than for generic net-graphs. Still, enhancement is not sufficient and, like in (3), such contributions remain statistically damped in naive continuum limit. Therefore transition from (3) to (14) should still be made “by hands”.

Related open question concerns generalization of background  $\hat{T}$  from the rank 3 case (related to associative algebras) to generic situation and connection of this problem to Batalin-Vilkovisky theory of Massis operations [17–19].

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