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Moduli integrals and ground ring in minimal liouville gravity

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Straightforward evaluation of the correlation functions in 2D minimal gravity requires integration over the moduli space. For degenerate fields the Liouville higher equations of motion allow to turn the integrand to a derivative and thus to reduce it to the boundary terms plus so-called curvature contribution. The last is directly related to the expectation value of the corresponding ground ring element. We use the operator product expansion technique to reproduce the ground ring construction explicitly in terms of the (generalized) minimal matter and Liouville degenerate fields. The action of the ground ring on the generic primary fields is evaluated explicitly. This permits us to construct directly the ground ring algebra. Detailed analysis of the ground ring mechanism is helpful in the understanding of the boundary terms and their evaluation.

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1.Introduction

1. Liouville gravity (LG) is the term for the two-dimensional quantum gravity whose action is induced by a "critical" matter, i.e., the matter described by a conformal field theory (CFT) $\mathcal{M}_c$ with central charge c. This induced action is universal and is called the Liouville action, because its variation with respect to the metric is proportional to the Liouville (or constant curvature) equation [1]. Let us denote $\{\Phi_i, \Delta_i\}$ be the set of primary fields and their dimensions in $\mathcal{M}_c$.

2. Liouville field theory (LFT) is constructed as the quantized version of the classical theory based on the Liouville action. LFT is again a conformal field theory with central charge $c_{\rm L}$. It is convenient to parameterize it in terms of variable b or $Q = b^{-1} + b$ as

$$c_{\rm L} = 1 + 6Q^2. \tag{1}$$

Parameter b enters the local Lagrangian

$$\mathcal{L}_{\rm L} = \frac{1}{4\pi}(\partial_a\phi)^2 + \mu e^{2b\phi} \tag{2}$$

where ϕ is the dynamical variable for the quantized metric ($ds^2 = \exp(2b\phi)\,\hat{g}_{ab}dx^a dx^b$ is interpreted as the metric in isothermal coordinate system[2)]) and μ is the scale parameter called the cosmological constant. Basic primary fields are the exponential operators $V_a = \exp(2a\phi)$, parameterized by a continuous (in general complex) parameter a in the way that the corresponding conformal dimension is $\Delta_a^{(\rm L)} = a(Q-a)$. Liouville field theory is exactly solvable [2]. In particular the three-point function $C^{(\rm L)}_{a_1,a_2,a_3} = \langle V_{a_1}(x_1)V_{a_2}(x_2)V_{a_3}(x_3)\rangle_{\rm L}$ is known explicitly for arbitrary exponential fields (see e.g. [3]). In LG the parameter b is chosen in the way that together with $\mathcal{M}_c$ LFT forms a joint conformal field theory with central charge $c + c_{\rm L} = 26$. Technically it is also convenient to include the

3. Reparametrization ghost field theory. This is the standard fermionic BC system of spin $(2,-1)$

$$A_{\rm gh} = \frac{1}{\pi}\int (C\bar{\partial}B + \bar{C}\partial\bar{B})d^2x \tag{3}$$

with central charge -26, which corresponds to the gauge fixing Faddeev-Popov determinant. The matter+Liouville stress tensor T generates a Virasoro algebra with central charge 26. Together with the ghost field theory this allows to form a BRST complex with respect to the nilpotent BRST charge

$$\mathcal{Q} = \oint (CT + C\partial CB)\,\frac{dz}{2\pi i}. \tag{4}$$

4. Correlation functions is one of the most important problems in the LG. In gravitational correlation functions the matter operators Φ_i are "dressed" by appropriate exponential Liouville fields V_{a_i} in the way to form either the $(1,1)$ form $U_i = \Phi_i V_{a_i}$ of ghost number 0 or the dimension $(0,0)$ operator $W_i = C\bar{C}U_i$ of ghost number 1. In both cases this requires

$$\Delta_i + a_i(Q - a_i) = 1. \tag{5}$$

Invariant (or integrated) correlation functions are independent on any coordinates and better called the correlation numbers. In the field theoretic framework, a

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[2)] $\hat{g}_{ab}$ is the "reference metric", a technical tool needed to give LFT a covariant form.

(genus 0) correlation number $\langle U_1 \ldots U_n \rangle_{\mathrm{G}}$ at $n \geq 3$ is constructed as the integral

$$\langle U_1 \ldots U_n \rangle_{\mathrm{G}} = \int_{M_n} \langle W_1(x_1) \ldots W(x_n) \rangle = \tag{6}$$
$$\int \langle W_1(x_1) W_2(x_2) W_3(x_3) U_4(x_4) d^2x_4 \ldots U(x_n) d^2x_n \rangle .$$

The integration here is over the moduli space M_n of the sphere with n punctures. Technically it is equivalent to choose any 3 of W_i at arbitrary fixed positions x_1, x_2 and x_3 and integrate the $(1,1)$ forms $U_i(x_i)d^2x_i$ inserted instead of W_i at $i = 4, \ldots, n$. At $n < 3$ the definition is slightly different. This is because of non-trivial conformal symmetries of the sphere with 2 and 0 punctures.

The simplest case of (6) is the tree-point function, where the moduli space is trivial and the result is factorized in a product of the matter, Liouville and ghost three-point functions

$$\langle U_1 U_2 U_3 \rangle_{\mathrm{G}} = x_{12}\bar{x}_{12}x_{23}\bar{x}_{23}x_{31}\bar{x}_{31} \times \tag{7}$$
$$\times \langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3) \rangle_{\mathrm{CFT}} \langle V_1(x_1)V_2(x_2)V(x_3) \rangle_{\mathrm{L}} .$$

The three-point functions $\langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3) \rangle_{\mathrm{CFT}}$, or the structure constants of the OPE algebra, are known explicitly in solvable matter CFT's. Thus eq.(7) gives the LG 3-point function in the explicit form. The two-point function and the zero-point one (the partition sum) are simply read off from this expression of the 3-point function.

5. The four-point function is the next step in the order of complexity

$$\langle U_1 U_2 U_3 U_4 \rangle_{\mathrm{G}} = x_{12}\bar{x}_{12}x_{23}\bar{x}_{23}x_{31}\bar{x}_{31} \times \tag{8}$$
$$\times \int \langle \Phi_1(x_1) \ldots \Phi_4(x_4) \rangle_{\mathrm{CFT}} \langle V_1(x_1) \ldots V_4(x_4) \rangle_{\mathrm{L}} \, d^2x_4.$$

This expression is much less explicit. First, it involves the integration over x_4. Then, even if the matter 4-point function is known in any convenient form, general representations for the Liouville four-point function are more complicated. E.g., the "conformal block" decomposition [3] involves the so called general conformal block [4], which is by itself a complicated function of its arguments, not to talk about the integration over the "intermediate momentum" P in [3]. In the present paper we take a preliminary step towards the evaluation of the four-point integral in the special case of

6. Minimal gravity (MG). If the conformal matter $\mathcal{M}_c$ is represented by a minimal CFT model (more precisely, a "generalized minimal model" (GMM), see below) $\mathcal{M}_{b^2}$, we talk about the "minimal gravity" (MG) (respectively generalized minimal gravity (GMG)). In GMG the evaluation of the four-point integral is dramatically simplified in the case when one of the matter operators Φ_i in the r.h.s. of (6) is a degenerate field $\Phi_{m,n}$. This is due to the so called "higher equations of motion" (HEM) which hold for the operator fields in LFT [5]. If $U_4 = U_{m,n} = \Phi_{m,n}\tilde{V}_{m,n}$ ($\tilde{V}_{m,n}$ is an appropriate dressing for $\Phi_{m,n}$), HEM allow to rewrite the integrand as a derivative and thus reduce the problem to boundary terms plus so-called curvature term. The last is directly expressed in terms of the expectation value $\langle O_{m,n} W_1 W_2 W_3 \rangle$ of the

7. Ground ring (GR) element $O_{m,n}$ related to the field $\Phi_{m,n}$. Therefore, we want to learn handle the ground ring algebra and the correlation functions of its elements. This knowledge will also prove instructive in the subsequent calculations of the boundary terms.

2. Generalized minimal models

Strictly speaking, minimal models of CFT $\mathcal{M}_{p/p'}$ [4] are consistently defined as a field theoretic constructions only if the "parameter" p/p' is an irreducible rational number so that p and p' are coprime integers. In this case the finite set of $(p-1)(p'-1)/2$ degenerate primary fields $\Phi_{m,n}$ with $1 \leq m < p$ and $1 \leq n < p'$ (modulo the identification $\Phi_{m,n} = \Phi_{p-m,p'-n}$) form, together with their irreducible representations, the whole space of $\mathcal{M}_{p/p'}$. "Canonical" minimal models $\mathcal{M}_{p/p'}$ are believed to be a completely consistent CFT, i.e., to satisfy all standard requirements of quantum field theory, except for the unitarity. They are also considered exactly solvable as the structure of their operator product expansion (OPE) algebra are known explicitly [6].

There are many ways to relax some of the requirements leading to the set of $\mathcal{M}_{p/p'}$ as unique CFT structures. For example, in the literature the "parameter" p/p' is often taken as an arbitrary number (e.g., [6]). The algebra of the degenerate primary fields doesn't close any more within any finite subset and rather the whole set $\{\Phi_{m,n}\}$ with (m,n) any natural numbers forms a closed algebra. Moreover, other authors include local fields with dimensions different from the Kac values, even continuous spectrum of dimensions. Although the consistency of such constructions from the field theoretic point of view remains to be clarified, these extensions prove to be a convenient technical tool. Moreover, statistical mechanics offers a number of examples where either a generalization of $\mathcal{M}_{p/p'}$ for non-integer p/p' is essentially necessary or non-degenerate primary operators appear as observables (or both).

In this paper we denote b^2 the parameter p/p' and admit the notion of GMM in the most wide sense as a conformal field theory with central charge

$$c = 1 - 6(b^{-1} - b)^2 \tag{9}$$

which may involve fields Φ_α of any dimension. Continuous parameter α is introduced to parameterize a continuous family of primary fields with dimensions $\Delta_\alpha = \alpha(\alpha - q)$, where $q = b^{-1} - b$. Also we always use the "canonical" CFT normalization of the primary fields Φ_α through the two-point functions $\langle \Phi_\alpha \Phi_\alpha \rangle_{\text{GMM}} = (x\bar{x})^{-2\Delta_\alpha}$.

Degenerate fields $\Phi_{m,n}$ have dimensions $\Delta^{(\text{M})}_{m,n} = -q^2/4 + \lambda^2_{m,-n}$, where a convenient notation

$$\lambda_{m,n} = mb^{-1}/2 + nb/2 \tag{10}$$

is introduced. They correspond to either $\alpha = \alpha_{m,n}$ or $\alpha = q - \alpha_{m,n}$ with $\alpha_{m,n} = q/2 + \lambda_{-m,n}$. The main restrictions, which singles out this apparently loose construction, is that the

1.Degenerate fields $\Phi_{1,2}$ and $\Phi_{2,1}$ (and therefore in general the whole set $\{\Phi_{m,n}\}$) are in the spectrum.

2. The null-vectors in the degenerate representations $\Phi_{m,n}$ vanish

$$D^{(\text{M})}_{m,n} \Phi_{m,n} = \bar{D}^{(\text{M})}_{m,n} \Phi_{m,n} = 0. \tag{11}$$

Here $D^{(\text{M})}_{m,n}$ ($\bar{D}^{(\text{M})}_{m,n}$) are the operators made of the right Virasoro generators M_n [3)] (respectively left $\bar{M}_n$) which create the level mn singular vector in the Virasoro module of $\Phi_{m,n}$. For definiteness we normalize these operators through the M_{-1}^{mn} term as $D^{(\text{M})}_{m,n} = M_{-1}^{mn} + \ldots$ First examples read explicitly

$$D^{(\text{M})}_{1,2} = M_{-1}^2 - b^2 M_{-2} \tag{12}$$
$$D^{(\text{M})}_{1,3} = M_{-1}^3 - 2b^2(M_{-2}M_{-1} + M_{-1}M_{-2}) + 4b^4 M_{-3}.$$

It turns out that this set of definitions imposes important restrictions on the structure of this formal construction. The three-point function $C_{\text{M}}(\alpha_1, \alpha_2, \alpha_3) = \langle \Phi_{\alpha_1} \Phi_{\alpha_2} \Phi_{\alpha_3} \rangle_{\text{GMM}}$ of "generic" primary fields can be restored uniquely from the above requirements [7]. At the degenerate values of the parameters $\alpha_i = \alpha_{m_i,n_i}$ (and if the standard "fusion" relations are satisfied) the known degenerate structure constants [6] are recovered.

Explicit form of the OPE of $\Phi_{1,2}$ and a generic primary field Φ_α

$$\Phi_{1,2}(x)\Phi_\alpha(0) = C^{(\text{M})}_+(\alpha)(x\bar{x})^{\alpha b}\left[\Phi_{\alpha+b/2}\right] + C^{(\text{M})}_-(\alpha)(x\bar{x})^{1-\alpha b - b^2}\left[\Phi_{\alpha-b/2}\right]$$

($[\Phi_\alpha]$ stands for a primary field Φ_α and all the tower of its conformal descendants) will be of use below. In our normalization

$$C^{(\text{M})}_+(\alpha) = \left[\frac{\gamma(b^2)\gamma(2\alpha b + 2b^2 - 1)}{\gamma(2b^2 - 1)\gamma(b^2 + 2\alpha b)}\right]^{1/2}, \tag{13}$$
$$C^{(\text{M})}_-(\alpha) = \left[\frac{\gamma(b^2)\gamma(2\alpha b + b^2 - 1)}{\gamma(2b^2 - 1)\gamma(2\alpha b)}\right]^{1/2}.$$

Other exact results in GMM form a somewhat miscellaneous collection. What is important for our program is the construction of the four-point function $\langle \Phi_{m,n}(x)\Phi_{\alpha_1}(x_1)\Phi_{\alpha_2}(x_2)\Phi_{\alpha_3}(x_3)\rangle_{\text{GMM}}$ with one degenerate field $\Phi_{m,n}$ and three generic primaries Φ_α[4]. The null-vector decoupling conditions (11) entail in general a system of partial differential equations. In the four-point case it reads as an ordinary linear differential equation of order mn, whose independent solutions are the conformal blocks $\mathcal{F}_{r,s}(x)$ appearing in this correlation function. The four-point function is then combined as

$$G^{(\text{GMM})}_{(m,n),\alpha_1,\alpha_2,\alpha_3}(x) = \sum_{r,s}^{(m,n)} C_{\text{M}}(\alpha_{m,n}, \alpha_1, \alpha_1 + \lambda_{r,-s}) \times C_{\text{M}}(\alpha_1 + \lambda_{r,-s}, \alpha_2, \alpha_3)\mathcal{F}_{r,s}(x)\mathcal{F}_{r,s}(\bar{x}), \tag{14}$$

where $\lambda_{r,s}$ are as in eq.(10) and the sign $\sum_{r,s}^{(m,n)}$ stands for the sum over the following set of integers (we use the notation $\{n_1 : d : n_1 + nd\} = \{n_1, n_1 + d, \ldots, n_1 + nd\}$)

$$(r, s) = (\{-m + 1 : 2 : m - 1\}, \{-n + 1 : 2 : n - 1\}). \tag{15}$$

Presently, when considering the GMG we restrict ourselves only to the four point function with one degenerate matter field $\Phi_{m,n}$, leaving the other three to be formal generics Φ_α. In particular, expression (14) is the relevant construction for the matter part of the integrand in eq.(8).

When dealing with GMM it is important to keep in mind that there are objects of different. Some are continuous in the parameter b^2, like the central charge, degenerate dimensions or certain correlation functions. Others may be highly discontinuous and dependent on the arithmetic nature of the numbers p and p'. Simplest example is the number of irreducible Virasoro representations entering the theory. This warns us to be careful when trying to reproduce the results of $\mathcal{M}_{p/p'}$ as a naive limit of $\mathcal{M}_{b^2}$ as $b^2 \to p/p'$ and $\alpha \to \alpha_{m,n}$ in the formal primary fields. This is why we stress that

[3)] Unusual notations M_n for the Virasoro generators of the matter conformal symmetry are chosen to save L_n for the generators of the Liouville Virasoro.

the three matter fields Φ_α in the matter function have generic non-degenerate values of the parameters α_1, α_2 and α_3.

3. Higher equations of motion

Let $a_{m,n} = Q/2 - \lambda_{m,n}$ with (m,n) a pair of positive integers, so that $V_{m,n} = V_{a_{m,n}}$ are the Liouville exponentials corresponding to degenerate representations of the Liouville Virasoro algebra. Let also $D^{(\mathrm{L})}_{m,n}$ be the corresponding "singular vector creating" operators made of the Liouville Virasoro generators L_n, similar to the operators $D^{(\mathrm{M})}_{m,n}$ introduced above. In fact $D^{(\mathrm{L})}_{m,n}$ is obtained from $D^{(\mathrm{M})}_{m,n}$ through the substitution $M_n \to L_n$ and $b^2 \to -b^2$. Like in GMM, in LFT the corresponding singular states vanish

$$D^{(\mathrm{L})}_{m,n} V_{m,n} = \bar{D}^{(\mathrm{L})}_{m,n} V_{m,n} = 0. \qquad (16)$$

Let $D^{(\mathrm{L})}_{m,n}$ be normalized similarly to (12) as $D^{(\mathrm{L})}_{m,n} = L^{mn}_{-1} + \ldots$

Define also the "logarithmic degenerate" fields

$$V'_{m,n} = \frac{1}{2}\frac{\partial}{\partial a} V_a|_{a=a_{m,n}} \qquad (17)$$

for every pair (m,n) of natural numbers. These fields are not primary. Under conformal transformations $x \to y$ they transform as

$$|y_x|^{2\Delta_{m,n}} V'_{m,n}(y) = V'_{m,n}(x) - \Delta'_{m,n} V_{m,n}(x) \log|y_x|, \qquad (18)$$

where y_x stands for $\partial y/\partial x$. Nevertheless, as it is shown in [5] $D^{(\mathrm{L})}_{m,n}\bar{D}^{(\mathrm{L})}_{m,n}V'_{m,n}$ is a primary field and, moreover, the following identity holds for the LFT operators

$$D^{(\mathrm{L})}_{m,n}\bar{D}^{(\mathrm{L})}_{m,n}V'_{m,n} = B_{m,n}\tilde{V}_{m,n}, \qquad (19)$$

where $\tilde{V}_{m,n} = V_a|_{a=a_{m,-n}}$ is the Liouville exponential of dimension $\Delta^{(\mathrm{L})}_{m,n} + mn$. The numerical constant $B_{m,n}$ reads

$$B_{m,n} = \frac{\left(\pi\mu\gamma(b^2)\right)^n b^{1+2n-2m}}{\gamma(1-m+nb^2)} \prod\nolimits_{k,l}^{\{m,n\}} 2\lambda_{k,l}, \qquad (20)$$

where $\prod_{k,l}^{\{m,n\}}$ stands for the product over $(k,l) = (\{-m+1:1:m-1\}\otimes\{-n+1:1:n-1\})\setminus(0,0)$. It is important to observe is that in GMG the exponential $\tilde{V}_{m,n}$ is naturally combined with the corresponding minimal matter field $\Phi_{m,n}$ to form the dressed $(1,1)$ form

$$U_{m,n} = \Phi_{m,n}\tilde{V}_{m,n}. \qquad (21)$$

This fact makes HEM crucial for the integrability of (8) in MG.

4. Generalized Minimal Gravity

Here we quote some known results in GMG. It is repeatedly observed in the literature, that in GMG the matter GMM parameter b coincides with the one of the corresponding LFT. This is why we keep the same notation throughout this paper. For the dressed matter fields $U_a = \Phi_\alpha V_a$, eq.(5) allows two solutions. For definiteness let's take

$$U_a = \Phi_{a-b} V_a. \qquad (22)$$

The GMG problem is to evaluate the gravitational correlation functions (6) with the matter part given by the GMM expressions. Thus in GMG we're restricted to the cases where the GMM correlation function is unambiguously determined.

The three-point function is easily calculated by multiplying $C_{\mathrm{M}}(a_1-b, a_2-b, a_3-b)$ by the corresponding Liouville three-point function $C^{(\mathrm{L})}_{a_1,a_2,a_3}$. The resulting product can be written in the form

$$\langle W_{a_1}W_{a_2}W_{a_2}\rangle_{\mathrm{GMG}} = \Omega N(a_1)N(a_2)N(a_3), \qquad (23)$$

where $W_a = C\bar{C}U_a$[4)],

$$\Omega = \left[\pi\mu\gamma(b^2)\right]^{Q/b}\left[\gamma(b^2)\gamma(b^{-2}-1)b^{-2}\right]^{1/2} \qquad (24)$$

and the "leg-factors" $N(a)$ read

$$N(a) = \left[\pi\mu\gamma(b^2)\right]^{-a/b}\left[\gamma(2ab-b^2)\gamma(2ab^{-1}-b^{-2})\right]^{1/2}. \qquad (25)$$

The two-point function $\langle U_aU_a\rangle_{\mathrm{GMG}}$ and the partition sum Z_{L} can be restored from this expression as

$$\langle U_aU_a\rangle_{\mathrm{GMG}} = \left[\pi\mu\gamma(b^2)\right]^{Q/b}\frac{N^2(a)}{\pi(2a-Q)} \qquad (26)$$

and

$$Z_{\mathrm{L}} = \left[\pi\mu\gamma(b^2)\right]^{Q/b}\frac{1-b^2}{\pi^3 Q\gamma(b^2)\gamma(b^{-2})}. \qquad (27)$$

For the normalized correlation functions $\langle\langle W_{a_1}W_{a_2}W_{a_2}\rangle\rangle = Z_{\mathrm{L}}^{-1}\langle W_{a_1}W_{a_2}W_{a_2}\rangle_{\mathrm{GMG}}$ and $\langle\langle U_aU_a\rangle\rangle = Z_{\mathrm{L}}^{-1}\langle U_aU_a\rangle_{\mathrm{GMG}}$ it is convenient to use slightly different leg-factors

$$\mathcal{N}(a) = \pi N(a)\left[\frac{\gamma(b^2)\gamma(b^{-2})}{-(b^{-2}-1)^2}\right]^{1/2}, \qquad (28)$$

4) Later on we'll use sometimes less compact notations $U(a) = U_a$ and $W(a) = W_a$.

so that

$$\langle\langle W_{a_1}W_{a_2}W_{a_2}\rangle\rangle = (1+b^{-2})b^{-2}(b^{-2}-1)\prod_{i=1}^{3}\mathcal{N}(a_i),$$
$$\langle\langle U_a U_a\rangle\rangle = \frac{(b^{-2}+1)b^{-2}(b^{-2}-1)}{(2ab^{-1}-b^{-2}-1)}\mathcal{N}^2(a). \quad (29)$$

At the generic values of a it will prove convenient to define renormalized fields

$$\mathcal{U}(a) = \mathcal{N}^{-1}(a)U_a\ ;\quad \mathcal{W}(a) = \mathcal{N}^{-1}(a)W_a, \quad (30)$$

for which (29) is reduced to

$$\langle\langle \mathcal{U}(a)\mathcal{U}(a)\rangle\rangle = \frac{(g+1)g(g-1)}{(2s-g-1)}, \quad (31)$$
$$\langle\langle \mathcal{W}(a_1)\mathcal{W}(a_2)\mathcal{W}(a_3)\rangle\rangle = (g+1)g(g-1),$$

where $g = b^{-2}$ and $s = ab^{-1}$. It is readily verified that formally $\mathcal{W}(a) = \mathcal{W}(Q-a)$, i.e., in this normalization the dressed matter operators are independent on the choice of the dressing. This might seem an important advantage. The price to pay is that the leg-factors (25) are sometimes singular and in any case depend on the cosmological constant μ.

5. Discrete states and the four-point integral

The next level of difficulty is the four-point correlation number $\langle U_{a_1}U_{a_2}U_{a_3}U_{a_4}\rangle_{\mathrm{GMG}}$ given by the integral (8). If one of the four matter operators, e.g., $\Phi_{\alpha_4} = \Phi_{m,n}$ the matter four-point function is constructed through (14). Let the rest three fields stay generic formal primaries of GMM[5]. Our purpose is to evaluate the integral

$$\langle U_{m,n}U_{a_1}U_{a_2}U_{a_3}\rangle_{\mathrm{GMG}} =$$
$$= \int \langle U_{m,n}(x)W_{a_1}(x_1)W_{a_2}(x_2)W_{a_3}(x_3)\rangle\, d^2x, \quad (32)$$

where $U_{m,n}$ is the dressed degenerate field $\Phi_{m,n}$ defined in (21). Denote

$$\Theta_{m,n} = \Phi_{m,n}V_{m,n} \quad (33)$$

the direct product of the matter and Liouville degenerate fields, and introduce the operators

$$\mathcal{D}_{m,n} = D^{(\mathrm{M})}_{m,n} + (-)^{mn}D^{(\mathrm{L})}_{m,n} \quad (34)$$

(and similarly $\bar{\mathcal{D}}_{m,n}$) where $D^{(\mathrm{M})}_{m,n}$ and $D^{(\mathrm{L})}_{m,n}$ are the matter and Liouville "singular vector creating" operators introduced above.

[5] As we'll discuss at the end of the paper, the last requirement is essential, because sometimes correlation functions with degenerate fields are not straightforward limits of those with generic ones with the appropriate specialization of the parameter.

Proposition 1: For every pair (m,n) of positive integers there exists an operator $H_{m,n}$, made of the Virasoro generators M_n, L_n and the ghost fields B and C as a graded polynomial of order $mn-1$ and ghost number 0, such that $H_{m,n}\Theta_{m,n}$ is closed but non-trivial. Operator $H_{m,n}$ is unique modulo exact terms.

Statement 1 can be verified by explicit calculations on the first levels. One finds

$$H_{1,2} = M_{-1} - L_{-1} + b^2 CB, \quad (35)$$
$$H_{1,3} = M_{-1}^2 - M_{-1}L_{-1} + L_{-1}^2 - 2b^2(M_{-2}+L_{-2}) +$$
$$+ 2b^2(M_{-1}-L_{-1})CB - 4b^4 C\partial B.$$

For the series $(1,n)$ a proof is given in ref.[8]. At general (m,n) the statement is most certainly also true (B.Feigin, private communication). Cohomology classes of $H_{m,n}\Theta_{m,n}$ were discovered in [9, 10] and are called the "discrete states". Although the generic form of the operators $H_{m,n}$ is not known to us, the normalization is supposed to be fixed as $H_{m,n} = \sum_{k=0}^{mn-1}(M_{-1})^{mn-1-k}(-L_{-1})^k + \ldots$. Apparently

$$(\partial H_{m,n} - \mathcal{Q}R_{m,n})\,\Theta_{m,n} = \left(\bar\partial \bar H_{m,n} - \bar{\mathcal{Q}}\bar R_{m,n}\right)\Theta_{m,n} = 0, \quad (36)$$

where $R_{m,n}$ is again a graded polynomial in M_n, L_n and ghosts.

Proposition 2:

$$\mathcal{D}_{m,n}\bar{\mathcal{D}}_{m,n}\Theta'_{m,n} =$$
$$= (\partial H_{m,n} - \mathcal{Q}R_{m,n})\left(\bar\partial \bar H_{m,n} - \bar{\mathcal{Q}}\bar R_{m,n}\right)\Theta'_{m,n}, \quad (37)$$

where $\Theta'_{m,n} = \Phi_{m,n}V'_{m,n}$ and $V'_{m,n}$ is from eq.(17)

We verified the Statement 2 directly for $(m,n) = (1,2)$ and $(1,3)$. Thus, general case might require modifications. Combined with HEM (19) it permits us to replace eq.(32) by

$$B^{-1}_{m,n}\int \partial\bar\partial\left\langle O'_{m,n}(x)W_{a_1}(x_1)W_{a_2}(x_2)W_{a_3}(x_3)\right\rangle d^2x, \quad (38)$$

where $O'_{m,n} = H_{m,n}\bar H_{m,n}\Theta'_{m,n}$. This is hence reduced to the boundary integral and the so-called curvature contribution. The boundary consists of small circles $\partial\Gamma_i$ around the W-insertions. To evaluate the boundary terms we need to understand better the short-distance behavior of the operator product $O'_{m,n}(x)W_a(x_1)$. Presently we discuss only the curvature term.

6. Curvature term

The curvature term comes from the fact that the operator $O'_{m,n}$ is not exactly a scalar ($(0,0)$ form) but a logarithmic field. Under conformal coordinate transformations $x \to y$ it acquires an inhomogeneous part

$$O'_{m,n}(y) = O'_{m,n}(x) - \Delta'_{m,n} O_{m,n}(x) \log |y_x| , \quad (39)$$

where

$$O_{m,n} = H_{m,n} \bar{H}_{m,n} \Theta_{m,n} \quad (40)$$

is the ground ring element (see below) and

$$\Delta'_{m,n} = \frac{d}{da} \Delta_a^{(\mathrm{L})} |_{a=a_{m,n}} = mb^{-1} + nb. \quad (41)$$

This subtlety is can be treated in two ways. First, it is easy to show that on the sphere the transformation (39) leads to the following behavior of the correlation function with $O'_{m,n}(x)$ at $x \to \infty$

$$\begin{aligned} &\langle O'_{m,n}(x) W_{a_1}(x_1) W_{a_2}(x_2) W_{a_3}(x_3) \rangle \sim \\ &\sim -\Delta'_{m,n} \log(x\bar{x}) \langle O_{m,n} W_{a_1} W_{a_2} W_{a_3} \rangle . \end{aligned}$$

Therefore the curvature contribution can be included as a boundary term $\partial\Gamma_\infty$ at ∞. Its is evaluated as

$$\begin{aligned} \frac{1}{2i} \int\limits_{\partial\Gamma_\infty} &\partial \langle O'_{m,n}(x) W_{a_1}(x_1) W_{a_2}(x_2) W_{a_3}(x_3) \rangle \, dx = \\ &= \pi \Delta'_{m,n} \langle O_{m,n} W_{a_1} W_{a_2} W_{a_3} \rangle . \end{aligned} \quad (42)$$

Another trick, which is easier generalized for more complicated surfaces, is to keep trace of the background metric $\widehat{g}_{ab} = e^\sigma \delta_{ab}$. Since the scale factor $\sigma(x)$ transforms as

$$\sigma(y) = \sigma(x) - 2 \log |y_x|$$

under conformal maps, the combination

$$\tilde{O}'_{m,n}(x) = O'_{m,n}(x) - \Delta'_{m,n} \sigma(x) O_{m,n}(x)/2 \quad (43)$$

is a scalar (the dependence on the background metric is the price to pay). Thus, in the BRST invariant environment equation (37) can be rewritten as

$$B_{m,n} U_{m,n} = \sqrt{\widehat{g}} \left(\frac{1}{4} \widehat{\triangle} \tilde{O}'_{m,n} - \frac{\Delta'_{m,n}}{8} \widehat{R} O_{m,n} \right) + \text{exact},$$

where $\widehat{\triangle}$ is the covariant Laplace operator with respect to $\widehat{g}_{ab}$ and $\widehat{R}$ is the corresponding scalar curvature. On a sphere the contribution of the second term apparently reduces to (42).

At this step it is clear that better understanding of the ground ring structure in GMG, in particular the evaluation of the expectation value in the right hand side of eq.(42), is of importance in the program.

7. Ground ring in GMG

It has been discovered in refs.[9, 10] that in MG the degenerate fields $\Phi_{m,n}$ of GMM, when combined with the degenerate exponentials $V_{m,n}$ of the corresponding LFT, give rise to non-trivial BRST closed operators (40) with ghost number 0 and conformal dimension $(0,0)$. Some of these operators were evaluated explicitly in [8]. The spatial derivatives $\partial O_{m,n}$ and $\bar{\partial} O_{m,n}$ are exact (36) and therefore the correlation functions of these discrete states in the BRST closed environment do not depend on their positions. Moreover, as the BRST cohomology classes they form a closed ring under the operator product expansion called the ground ring. This observation led E.Witten [10] to conclude that this structure plays a crucial role in the structure of MG and probably the whole algebraic structure of the theory is in fact that of the ground ring. In this section we present few explicit calculations revealing the GR properties. Cohomology properties of $O_{m,n}$ are relevant only in a $\mathcal{Q}$-invariant environment. The simplest invariant state on a sphere is created by three operators W_a. For this reason we perform actual calculation of the correlation function of $\langle O_{m,n} W_{a_1} W_{a_2} W_{a_3} \rangle$ on a sphere with three generic W_a insertions.

Modulo exact forms the discrete states $O_{m,n}$ act in the space of classes W_a. This is because their action doesn't change the ghost number and generically all non-trivial classes are exhausted by the composite fields W_a with different a. Moreover, due to the decoupling restrictions in the OPE of the degenerate fields $\Phi_{m,n}$ and $V_{m,n}$ with the primaries Φ_α and V_a respectively, the general structure of the operator product $O_{m,n}(x) W(a)$ is doomed to have the form

$$O_{m,n} W(a) = \sum_{r,s}^{(m,n)} A_{r,s}^{(m,n)} W(a + \lambda_{r,s}) + \text{exact}, \quad (44)$$

with some numerical coefficients $A_{r,s}^{(m,n)}$. Our first aim is to evaluate them.

It is instructive to perform explicit calculations in the simplest case $(m,n) = (1,2)$. The special operator product expansions we need in this case are (13) and

$$\begin{aligned} V_{1,2}(y) V_a(0) = C_+^{(\mathrm{L})}(a) (y\bar{y})^{ab} \left[V_{a-b/2} \right] + \\ + C_-^{(\mathrm{L})}(a) (y\bar{y})^{1-ab+b^2} \left[V_{a+b/2} \right] , \end{aligned}$$

where

$$C_+^{(\mathrm{L})}(a) = 1 \, ; \quad C_-^{(\mathrm{L})}(a) = -\frac{\pi\mu}{\gamma(-b^2)} \frac{\gamma(2ab - b^2 - 1)}{\gamma(2ab)} .$$

It is easy to verify by explicit calculation (at least at the primary field level) that in the product $U_a = \Phi_{a-b} V_a$ the action of $H_{1,2}$ and $\bar{H}_{1,2}$ eliminates the "wrong terms" with the combinations $\Phi_{a-b/2}V_{a-b/2}$ and $\Phi_{a-3b/2}V_{a+b/2}$ and we are left with

$$O_{1,2}W(a) = A^{(1,2)}_{0,-1}W(a-b/2) + \\ + A^{(1,2)}_{0,1}W(a+b/2) + \text{exact}, \tag{45}$$

with

$$A^{(1,2)}_{0,-1} = \left(1-2ab+b^2\right)^2 C^{(\mathrm{M})}_{-}(a-b)C^{(\mathrm{L})}_{+}(a), \tag{46}$$
$$A^{(1,2)}_{0,1} = \left(1-2ab+b^2\right)^2 C^{(\mathrm{M})}_{+}(a-b)C^{(\mathrm{L})}_{-}(a).$$

The polynomial multipliers in the coefficients are the result of the action of $H_{1,2}$ on the corresponding terms in the expansion of $\Theta_{1,2}(x)W_a(0)$. Similar calculation can be directly performed for the action of every $\Phi_{m,n}$. We calculated these polynomials also for the case $(m,n)=(1,3)$ and verified that the result is summarized as follows

$$N(a+\lambda_{r,s})A^{(m,n)}_{r,s} = \Lambda_{m,n}N(a), \tag{47}$$

where

$$\Lambda_{m,n} = \left(\gamma(b^2)\gamma(b^{-2})(b-b^{-1})^{-2}\right)^{1/2} B_{m,n}N(a_{m,-n}) \tag{48}$$

and $B_{m,n}$ are the same as in eq.(20).

It seems tempting to simplify these relations by introducing the renormalized fields $\mathcal{W}(a)$ as in eq.(30) and $\mathcal{O}_{m,n} = \Lambda^{-1}_{m,n}O_{m,n}$ so that (44) is reduced to

$$\mathcal{O}_{m,n}\mathcal{W}(a) = \sum_{r,s}^{(m,n)} \mathcal{W}(a+\Lambda_{r,s}). \tag{49}$$

8.‘Discussion

This is basically what has been figured out previously on the basis of more general arguments [11]. Here we arrive at this expression by a direct calculation. Another important difference is that we considered the action of $O_{m,n}$ on a cohomology W_a with generic a. It is natural to expect, that the relations (49) are modified when specialized to the degenerate fields $W_{m,n} = C\bar{C}\Phi_{m,n}\tilde{V}_{m,n}$ with vanishing null vector in the degenerate matter sector. Although the effect most probably might be simply the proper truncation of the sum in eq.(49) implied by the fusion algebra of the degenerate fields, technically the limit $a \to a_{m,-n}$ in this expression turns out to be subtle and requires more careful analysis. Therefore, in this article we restrict ourselves with the case of generic values of a leaving the degenerate cases for further study. This is basically sufficient to our subsequent treatment of the integral (32) with generic non-degenerate values of a_1, a_2 and a_3.

The simple action (49) naturally implies the following structure of the ground ring algebra

$$\mathcal{O}_{m,n}\mathcal{O}_{m',n'} = \sum_{l}^{[m,m']}\sum_{k}^{[n,n']} \mathcal{O}_{l,k}, \tag{50}$$

where the symbol $\sum_k^{[n,n']}$ implies the sum over $k = \{\min(|n-n'|,0):2:n+n'\}$. Or, if you like better to follow [11] and introduce $X=\mathcal{O}_{1,2}/2$ and $Y=\mathcal{O}_{2,1}/2$

$$\mathcal{O}_{m,n} = U_{m-1}(Y)\,U_{n-1}(X), \tag{51}$$

where $U_n(x)$ are the Chebyshev polynomials of the second kind.

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