

SPECTRA OF RANDOM CONTRACTIONS AND SCATTERING THEORY FOR DISCRETE-TIME SYSTEMS

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Random contractions (sub-unitary random matrices) appear naturally when considering quantized chaotic maps within a general theory of open linear stationary systems with discrete time. We analyze statistical properties of complex eigenvalues of generic $N \times N$ random matrices \hat{A} of such a type, corresponding to systems with broken time-reversal invariance. Deviations from unitarity are characterized by rank $M \leq N$ and a set of eigenvalues $0 < T_i \leq 1, i = 1, \dots, M$ of the matrix $\hat{T} = \hat{1} - \hat{A}^\dagger \hat{A}$. We solve the problem completely by deriving the joint probability density of N complex eigenvalues and calculating all n -point correlation functions. In the limit $N \gg M, n$ the correlation functions acquire the universal form found earlier for weakly non-Hermitian random matrices.

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The theory of wave scattering can be looked at as an integral part of the general theory of linear dynamic open systems in terms of the input-output approach. These ideas and relations were developed in system theory and engineering mathematics many years ago, see papers [1, 2] and references therein. Unfortunately, that development went almost unnoticed by the majority of physicists working in the theory of chaotic quantum scattering and related phenomena, see [3, 4] and references therein. For this reason we feel it could be useful to recall some basic facts of the input-output approach in such a context.

An Open Linear System is characterized by three Hilbert spaces: the space E_0 of internal states $\Psi \in E_0$ and two spaces E_\pm of incoming (-) and outgoing (+) signals or waves also called input and output spaces, made of vectors $\phi_\pm \in E_\pm$. Acting in these three spaces are four operators, or matrices: a) the so-called fundamental operator \hat{A} which maps any vector from internal space E_0 onto some vector from the same space E_0 b) two operators $\hat{W}_{1,2}$, with \hat{W}_1 mapping incoming states onto an internal state and \hat{W}_2 mapping internal states onto outgoing states and c) an operator \hat{S}_0 acting from E_- to E_+ .

We will be interested in describing the dynamics $\Psi(t)$ of an internal state with time t provided we know the state at initial instant $t = 0$ and the system is subject to a given input signal $\phi_-(t)$. In what follows we consider only the case of the so-called stationary (or *time-invariant*) systems when the operators are assumed to be time-independent. Let us begin with the case of continuous-time description. The requirements of linearity, causality and stationarity lead to a system of two dynamical equations:

$$i \frac{d}{dt} \Psi = \hat{A} \Psi(t) + \hat{W}_1 \phi_-(t) \quad \text{and} \quad \phi_+(t) = \hat{S}_0 \phi_-(t) + i \hat{W}_2 \Psi(t). \quad (1)$$

Interpretation of these equations depends on the nature of the state vector Ψ as well as of the vectors ϕ_{\pm} and is different in different applications. In the context of quantum mechanics one relates the scalar product $\Psi^{\dagger}\Psi$ with the probability to find a particle inside the "inner" region at time t , whereas $\phi_{\pm}^{\dagger}\phi_{\pm}$ stays for probability currents flowing in and out of the region of internal states (the number of particles coming or leaving the inner domain per unit time). The condition of particle conservation then reads as:

$$\frac{d}{dt}\Psi^{\dagger}\Psi = \phi_{-}^{\dagger}\phi_{-} - \phi_{+}^{\dagger}\phi_{+}. \quad (2)$$

It is easy to verify that Eq.(2) is compatible with the dynamics Eq.(1) only provided the operators satisfy the following relations:

$$\hat{A}^{\dagger} - \hat{A} = i\widehat{W}\widehat{W}^{\dagger}, \quad \hat{S}_0^{\dagger}\hat{S}_0 = \hat{1} \quad \text{and} \quad \widehat{W}^{\dagger} \equiv \widehat{W}_2 = -\hat{S}_0\widehat{W}_1^{\dagger}$$

which shows, in particular, that \hat{A} can be written as $A = \hat{H} - (i/2)\widehat{W}\widehat{W}^{\dagger}$, with a Hermitian $\hat{H} = \hat{H}^{\dagger}$.

The meaning of \hat{H} is transparent: it governs the evolution $i(d/dt)\Psi = \hat{H}\Psi(t)$ of an inner state Ψ when the coupling \widehat{W} between the inner space and input/output spaces is absent. As such, it is just the Hamiltonian describing the closed inner region. The fundamental operator \hat{A} then has a natural interpretation of the effective non-selfadjoint Hamiltonian describing the decay of the probability from the inner region at zero input signal: $\phi_{-}(t) = 0$ for any $t \geq 0$. If, however, the input signal is given in the Fourier-domain by $\phi_{-}(\omega)$, the output signal is related to it by:

$$\phi_{+}(\omega) = \left[\hat{S}(\omega)\hat{S}_0 \right] \phi_{-}(\omega) \quad , \quad \hat{S}(\omega) = \hat{1} - i\widehat{W}^{\dagger} \frac{1}{\omega\hat{1} - \hat{A}} \widehat{W} \quad (3)$$

where we assumed $\Psi(t=0) = 0$. The unitary matrix $\hat{S}(\omega)$ is known in the mathematical literature as the characteristic matrix-function of the non-Hermitian operator \hat{A} . In the present context it is just the scattering matrix whose unitarity is guaranteed by the conservation law Eq(2).

The contact with the theory of chaotic scattering is now apparent: the expression Eq.(3) was frequently used in the physical literature [3, 4] as a starting point for extracting universal properties of the scattering matrix for a quantum chaotic system within the so-called random matrix approach. The main idea underlying such an approach is to replace the actual Hamiltonian \hat{H} by a large random matrix and to calculate the ensuing statistics of the scattering matrix. The physical arguments in favor of such a replacement can be found in the cited literature.

In particular, most recently the statistical properties of complex eigenvalues of the operator \hat{A} were studied in much detail [4]. Those eigenvalues are poles of the scattering matrix and have the physical interpretation of *resonances* - long-lived intermediate states to which discrete energy levels of the closed system are transformed due to coupling to continua.

In the theory presented above the time t was a continuous parameter. On the other hand, a very useful instrument in the analysis of classical Hamiltonian systems with chaotic dynamics are the so-called area-preserving chaotic maps [5]. They appear naturally either as a mapping of the Poincaré section onto itself, or as result of a stroboscopic description of Hamiltonians which are periodic functions of time. Their quantum

mechanical analogues are unitary operators which act on Hilbert spaces of finite large dimension N . They are often referred to as evolution, scattering or Floquet operators, depending on the physical context where they are used. Their eigenvalues consist of N points on the unit circle (eigenphases). Numerical studies of various classically chaotic systems suggest that the eigenphases conform statistically quite accurately the results obtained for unitary random matrices of a particular symmetry (Dyson circular ensembles).

Let us now imagine that a system represented by a chaotic map ("inner world") is embedded in a larger physical system ("outer world") in such a way that it describes particles which can come inside the region of chaotic motion and leave it after some time. Models of such type appeared, for example, in [6] where a kind of scattering theory for "open quantum maps" was developed based on a variant of Lipmann-Schwinger equation.

On the other hand, in the general system theory [2] dynamical systems with discrete time are considered as frequently as those with continuous time. For linear systems a "stroboscopic" dynamics is just a linear map $(\phi_-(n); \Psi(n)) \rightarrow (\phi_+(n); \Psi(n+1))$ which can be generally written as:

$$\begin{pmatrix} \Psi(n+1) \\ \phi_+(n) \end{pmatrix} = \hat{V} \begin{pmatrix} \Psi(n) \\ \phi_-(n) \end{pmatrix}, \quad \hat{V} = \begin{pmatrix} \hat{A} & \hat{W}_1 \\ \hat{W}_2 & \hat{S}_0 \end{pmatrix}. \quad (4)$$

Again we would like to consider a conservative system, and the discrete-time analogue of Eq.(2), in fact, requires the unitarity of the matrix \hat{V} in Eq.(4). It turns out that such a matrix \hat{V} can always be conveniently parametrized as (cf.[7]):

$$\hat{V} = \begin{pmatrix} \hat{u}\sqrt{1-\hat{\tau}\hat{\tau}^\dagger} & -\hat{u}\hat{\tau} \\ \hat{\tau}^\dagger & \sqrt{1-\hat{\tau}\hat{\tau}^\dagger} \end{pmatrix} \quad (5)$$

where \hat{u} is unitary and $\hat{\tau}$ is a rectangular $N \times M$ diagonal matrix with the entries $\tau_{ij} = \delta_{ij}\tau_j$, $1 \leq i \leq N$, $1 \leq j \leq M$, $0 \leq \tau_j \leq 1$. Here we actually used a freedom to redefine input, output and internal state by choosing an appropriate basis in the corresponding spaces. This allowed us to bring Eq.(5) to the simplest form suggesting a clear interpretation of the constituents of the model. Indeed, for $\hat{\tau} = 0$ the dynamics of the system amounts to: $\Psi(n+1) = \hat{u}\Psi(n)$. We therefore identify \hat{u} as a unitary evolution operator of the "closed" inner state domain decoupled both from input and output spaces. Correspondingly, $\hat{\tau} \neq 0$ just provides a coupling that makes the system open and converts the fundamental operator $\hat{A} = \hat{u}\sqrt{1-\hat{\tau}\hat{\tau}^\dagger}$ to a contraction: $1 - \hat{A}^\dagger\hat{A} = \tau\tau^\dagger \geq 0$. As a result, the equation $\Psi(n+1) = \hat{A}\Psi(n)$ describes an irreversible decay of any initial state $\Psi(0)$ for zero input $\phi_-(n) = 0$, whereas for a nonzero input and $\Psi(0) = 0$ the Fourier-transforms $\phi_\pm(\omega) = \sum_{n=0}^{\infty} e^{in\omega} \phi_\pm(n)$ are related by a unitary scattering matrix $\hat{S}(\omega)$ given by:

$$\hat{S}(\omega) = \sqrt{1-\hat{\tau}^\dagger\hat{\tau}} - \hat{\tau}^\dagger \frac{1}{e^{-i\omega} - \hat{A}} \hat{u}\hat{\tau}. \quad (6)$$

Assuming further that the motion outside the inner region is regular, we should be able to describe generic features of open quantized chaotic maps choosing the matrix \hat{u} to be a member of a Dyson circular ensemble. Then one finds: $\hat{\tau}^\dagger\hat{\tau} = 1 - \overline{|\hat{S}(\omega)|^2}$, with the bar standing for the averaging of $\hat{S}(\omega)$ in Eq.(6) over \hat{u} . Comparing this result with [3, 4] we see that the M eigenvalues $0 \leq T_i \leq 1$ of the $M \times M$ matrix $\hat{\tau}^\dagger\hat{\tau}$ play the role

of the so-called transmission coefficients and describe a particular way the chaotic region is coupled to the outer world.

In fact, this line of reasoning is motivated by recent papers [8, 9]. The authors of [8] considered the Floquet description of a Bloch particle in a constant force and periodic driving. After some approximations the evolution of the system is described by a mapping: $\mathbf{c}_{n+1} = \mathbf{F}\mathbf{c}_n$, where the unitary Floquet operator $\mathbf{F} = \widehat{S}\widehat{U}$ is the product of a unitary “ M -shift” $\widehat{S} : S_{kl} = \delta_{l, k-M}$, $l, k = -\infty, \dots, \infty$ and a unitary matrix \widehat{U} . The latter is effectively of the form $\widehat{U} = \text{diag}(\widehat{d}_1, \widehat{u}, \widehat{d}_2)$, where $\widehat{d}_{1,2}$ are (semi)infinite diagonal matrices and \widehat{u} can be taken from the ensemble of random $N \times N$ unitary matrices.

One can check that such a dynamics can be easily brought to the standard Eqs.(4), (5) with the fundamental operator being an $N \times N$ random matrix of the form $\widehat{A} = \sqrt{1 - \widehat{\tau}^\dagger \widehat{\tau}} \widehat{u}$, and all M diagonal elements of the $N \times M$ matrix τ are equal to unity. Actually, the original paper [8] employed a slightly different but equivalent construction dealing with an “enlarged” internal space of the dimension $N + M$. We prefer to follow the general scheme because of its conceptual clarity.

Direct inspection immediately shows that the non-vanishing eigenvalues of the fundamental operator \widehat{A} as above coincide with those of a $(N - M) \times (N - M)$ subblock of the random unitary matrix u . Complex eigenvalues of such “truncations” of random unitary matrices were studied in much detail by the authors of a recent insightful paper [9]. They managed to study eigenvalue correlations analytically for arbitrary N, M . In particular, they found that in the limit $N \rightarrow \infty$ for fixed M these correlation functions practically coincide [8] with those obtained earlier [4] for operators of the form $\widehat{A} = \widehat{H} - (i/2)\widehat{W}\widehat{W}^\dagger$ occurring in the theory of open systems with continuous-time dynamics.

Such a remarkable universality, though not completely unexpected, deserves to be studied in more detail. In fact, truncated unitary matrices represent only a particular case of random contractions \widehat{A} . Actually, some statistical properties of general subunitary matrices were under investigation recently as a model of scattering matrix for systems with absorption, see [10].

The particular case of rank-one deviations from unitarity is the simplest one to investigate and was considered in a recent preprint [11]. However, generalization to arbitrary M along the lines of [11] seemed to be problematic. The main goal of the present paper is to provide results for the spectra of random contractions for an arbitrary, given deviation from unitarity.

The ensemble of general $N \times N$ random contractions $\widehat{A} = \widehat{u}\sqrt{1 - \widehat{\tau}^\dagger \widehat{\tau}}$ describing the chaotic map with broken time-reversal symmetry can be described by the following probability measure in the matrix space:

$$\mathcal{P}(\widehat{A})d\widehat{A} \propto \delta(\widehat{A}^\dagger \widehat{A} - \widehat{G})d\widehat{A} \quad , \quad \widehat{G} \equiv 1 - \widehat{\tau}^\dagger \widehat{\tau} \quad , \quad (7)$$

where $d\widehat{A} = \prod_{i,j} d\widehat{A}_{ij} dA_{ij}^*$ and we assumed that the unitary matrix \widehat{u} is taken from the Dyson circular unitary ensemble. The $N \times N$ matrix $\widehat{\tau}^\dagger \widehat{\tau} = 1 - \widehat{G} \geq 0$ is natural to call the deviation matrix and we denote it \widehat{T} . It has M nonzero eigenvalues coinciding with the transmission coefficients T_a introduced above. The particular choice $T_{i \leq M} = 1$, $T_{i > M} = 0$ corresponds to the case considered in [9].

Our first step is, following [4, 9, 11], introduce the Schur decomposition $\widehat{A} = \widehat{U}(\widehat{Z} + \widehat{R})\widehat{U}^\dagger$ of the matrix A in terms of a unitary \widehat{U} , diagonal matrix of the eigenvalues \widehat{Z} and an upper triangular \widehat{R} . One can satisfy oneself, that the eigenvalues z_1, \dots, z_N are

generically not degenerate, provided all $T_i < 1$. Then, the measure written in terms of new variables is given by $d\hat{A} = |\Delta(\{z\})|^2 d\hat{R} d\hat{Z} d\mu(U)$, where the first factor is just the Vandermonde determinant of eigenvalues z_i and $d\mu(U)$ is the invariant measure on the unitary group. The integration over \hat{R} can be performed with some manipulations using its triangularity (some useful hints can be found in [9]). As the result, we arrive at the joint probability density of all eigenvalues $\mathcal{P}(\{z\})$ to be given by:

$$\mathcal{P}(\{z\}) \propto |\Delta(\{z\})|^2 \int d\mu(U) \prod_{l=1}^N \delta \left(|z_1|^2 \dots |z_l|^2 - \det \left[1 - \hat{T} \hat{U} \hat{P}_l \hat{U}^\dagger \right] \right), \quad (8)$$

where $\hat{P} = \text{diag}(1, \dots, 1, 0, \dots, 0)$, with first l entries being equal to unity, is a projector.

To perform the remaining integration over the unitary group was mentioned as the main technical problem in [11]. Relegating details of the calculation to a more extended publication we mention that integration can be performed in a recursive way $l \rightarrow l + 1$ with the multiple use of the Itzykson - Zuber - Harish - Chandra [12] formula. After quite an elaborate manipulation, one finally arrives at the following representation:

$$\begin{aligned} \mathcal{P}(\{z\}) \propto & \frac{\det^{M-N}(\hat{T})}{\det(1 - \hat{T}) \prod_{c_1 < c_2} (T_{c_1} - T_{c_2})} \prod_{c_1 < c_2} \left(\frac{\partial}{\partial \tau_{c_1}} - \frac{\partial}{\partial \tau_{c_2}} \right) \times \\ & \times \int d\hat{\lambda} e^{-i \text{Tr} \hat{\tau} \hat{\lambda}} |\Delta(\{z\})|^2 \prod_{k=1}^N f(\ln |z_k|^2, \hat{\lambda}), \end{aligned} \quad (9)$$

where we defined the diagonal matrices of size M as: $\hat{\tau} = \text{diag}(\tau_1, \dots, \tau_M)$, $\hat{\lambda} = \text{diag}(\lambda_1, \dots, \lambda_M)$ and used the notations: $\tau_c = \ln(1 - T_c)$ and

$$f(a, \hat{\lambda}) = i^{M-1} \sum_{q=1}^M \frac{e^{i\lambda_q a}}{\prod_{s(\neq q)} (\lambda_q - \lambda_s)}. \quad (10)$$

The distribution Eq.(9) is written for $|z_k|^2 \leq 1$ for any $k = 1, \dots, N$ and vanishes otherwise. The remarkable feature of such a distribution is that it allows for calculation of all n -point correlation functions for arbitrary N, n, M with help of the method of orthogonal polynomials. Again, the particular case $M = 1$ [11] is quite instructive and can be recommended to follow first for understanding of the general formulae outlined below.

To this end, we write

$$|\Delta(\{z\})|^2 \prod_{k=1}^N f(\ln |z_k|^2, \hat{\lambda}) = \prod_{k=1}^N N_k(\hat{\lambda}) \det \left[\sum_{n=1}^N \frac{(z_i z_j^*)^{n-1}}{N_n(\hat{\lambda})} f(\ln |z_j|^2, \hat{\lambda}) \right]_{i,j=1, \dots, N}, \quad (11)$$

where the constants $N_n(\hat{\lambda})$ are provided by the orthonormality condition:

$$\int_{|z|^2 \leq 1} d^2 z z^{m-1} (z^*)^{n-1} f(\ln |z|^2, \hat{\lambda}) = \delta_{m,n} N_n(\hat{\lambda}), \quad (12)$$

which yields after a simple calculation

$$N_n(\hat{\lambda}) = \pi \prod_{c=1}^M \frac{1}{(n + i\lambda_c)}.$$

Now, by applying the standard machinery of orthogonal polynomials [13] one can find the correlation function:

$$R_n(z_1, \dots, z_n) = \frac{N!}{(N-n)!} \int d^2 z_{n+1} \dots d^2 z_N \mathcal{P}(\{z\}) \quad (13)$$

as given by:

$$R_n(z_1, \dots, z_n) \propto \widehat{\mathcal{D}} \int d\widehat{\lambda} e^{-i \text{Tr} \widehat{\tau} \widehat{\lambda}} \prod_{k=1}^N N_k(\widehat{\lambda}) \det \left[K(z_i, z_j; \widehat{\lambda}) \right]_{(i,j)=1, \dots, n}, \quad (14)$$

where the kernel K is defined as:

$$K(z_1, z_2; \widehat{\lambda}) = \frac{1}{\pi} \sum_{n=1}^N \det(i\widehat{\lambda} + n) (z_1 z_2^*)^{n-1} f(\ln |z_2|^2, \widehat{\lambda}) \quad (15)$$

and the differential operator $\widehat{\mathcal{D}}$ is just the expression in front of the λ -integral in Eq.(9).

In principle, all λ -integrations in the equation Eq.(14) can be performed explicitly and the resulting formulae provide the desired general solution of the problem. However, for arbitrary N, M, n the results obtained in that way are still quite cumbersome. We present below as an example the lowest correlation function $R_1(z)$ (which is just the mean eigenvalue density inside the unit circle $|z| < 1$) for the case of all equivalent channels, i.e. when all the transmission coefficients T_c are equal: $T_c = T$. Such a function satisfies the recursive relation:

$$R_1^{(M)}(z) = R_1^{(M-1)}(z) + \frac{1}{(M-1)!} \left(\frac{\partial}{\partial t} \right)^{M-1} \mathcal{R}(t, z)|_{t=0}, \quad (16)$$

where the generating function is given by:

$$\mathcal{R}(t, z) = \frac{1}{\pi} \frac{\partial}{\partial |z|^2} \frac{\xi^N - \eta^N}{\xi - \eta}$$

and

$$\xi = 1 + (t-1) \left(\frac{1}{|z|^2} - 1 \right) \left(\frac{1}{T} - 1 \right), \quad \eta = 1 + (t-1) (1 - |z|^2) \frac{1}{T}. \quad (17)$$

All the equations above are valid for arbitrary $N \geq M, n$. In the theory of quantum chaotic scattering we, however, expect a kind of universality in the semiclassical limit. Translated to the random matrix language such a limit corresponds to $N \rightarrow \infty$ at fixed n, M . Simplifications occur after taking into account that eigenvalues z_i are, in fact, concentrated typically at distances of order of $1/N$ from the unit circle. Then it is natural to introduce new variables y_i, ϕ_i according to $z_i = (1 - y_i/N) e^{i\phi_i}$ and consider y_i to be of the order of unity when $N \rightarrow \infty$. As to the phases ϕ_i , we expect their typical separation scaling as: $\phi_i - \phi_j = O(1/N)$. Performing now explicitly the limit $N \rightarrow \infty$ one can bring the correlation function Eq.(14) after some lengthy manipulations to the form:

$$R_n(z_1, \dots, z_n) \propto \prod_{k=1}^n \sum_{q=1}^M \frac{e^{-g_q y_k}}{\prod_{s \neq q} (g_q - g_s)} \det \left[\int_{-1}^1 d\lambda \prod_{c=1}^M (\lambda + g_c) e^{-\frac{i}{2} \lambda \delta_{ij}} \right]_{i,j=1, n} \quad (18)$$

with $g_c = 2/T_c - 1$ and $\delta_{ij} = N(\phi_i - \phi_j) - i(y_i + y_j)$. The expression above coincides in every detail with that obtained in [4] for random GUE matrices deformed by a finite rank anti-Hermitian perturbation and thus provides the proof of universality for finite-rank deviations.

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1. M.S.Livsic, *Operators, Oscillations, Waves: Open Systems*, Translations of Mathematical Monographs, v.34, AMS, Providence, RI, 1973.
 2. D.Z.Arov, *Sib. Math. J.* **20**, 149 (1979); J.W.Helton, *J. Funct. Anal.* **16**, 15 (1974).
 3. C.H.Lewenkopf and H.A.Weidenmüller, *Ann. Phys. NY* **212**, 53 (1991).
 4. Y.V.Fyodorov and H.-J.Sommers, *J. Math. Phys.* **38**, 1918 (1997); Y.V.Fyodorov and B.A.Khoruzhenko, *Phys. Rev. Lett.* **83**, 65 (1999); H.-J.Sommers, Y.V.Fyodorov, and M.Titov, *J. Phys.* **A32**, L77 (1999).
 5. U.Smilansky, in *Supersymmetry and Trace Formulae: Chaos and Disorder*, Eds. I.V.Lerner et al., Kluwer, NY, 1999, p.173.
 6. F.Borgonovi, I.Guarneri, and D.L.Shepelyansky, *Phys. Rev.* **A43**, 4517 (1991).
 7. P.A.Mello and J.-L.Pichard, *J. de Phys. I* **1**, 493 (1991).
 8. M.Glück, A.R.Kolovski, and H.J.Korsch, *Phys. Rev.* **E60**, 247 (1999).
 9. K.Zyczkowski and H.-J.Sommers, *J. Phys. A: Math. Gen.* **33**, 2045 (2000).
 10. E. Kogan, P.A.Mello, and He Liqun, *Phys. Rev.* **E61**, R17 (2000); C.W.J.Beenakker and P.W.Brouwer, e-preprint cond-mat/9908325.
 11. Y.V.Fyodorov, e-preprint nlin.CD/0002034 at xxx.lanl.gov, to appear at: *Disordered and Complex Systems*, AIP Proceedings, 2001.
 12. C.Itzykson and J.B.Zuber, *J. Math. Phys.* **21**, 411 (1980); Harish-Chandra, *Am. J. Math.* **79**, 87 (1957).
 13. M.L.Mehta, *Random Matrices*, 2-nd ed., Academic Press, London, 1991.