

SOME EXACT SOLUTIONS FOR THE CLASSICAL HALL EFFECT IN INHOMOGENEOUS MAGNETIC FIELD

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The classical Hall effect in inhomogeneous systems is considered for the case of one-dimensional inhomogeneity. For a certain geometry of the problem and for the magnetic field linearly depending on the coordinate the density of current distribution corresponds to the skin-effect.

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The behaviour of 2D electrons in a spatially nonuniform magnetic field is of interest in various aspects. The simplest situation is one-dimensional inhomogeneity when the normal component of the magnetic field B_z varies only in one direction, say $B_z(y)$. Müller [1] has considered the ballistic regime of a 2D electronic system for a special case $B_z(y) = B_0ky$ and for the geometry when the current flows in the x -direction perpendicularly to the magnetic field gradient. Numerical solution of the Schrödinger equation carried out in [1] gives the current distribution in the y -direction $j_x(y)$. Nonzero values of $j_x(y)$ (without electric field) arise because the Landau degeneracy is lifted in the inhomogeneous magnetic field and the eigenstates become current-carrying. Of course, the total current $J = \int j_x(y)dy$ equals zero in the absence of the electric field (the case with electric field has not been considered in [1]).

In the present paper I will consider the classical magnetotransport for which the local relation between the current density \mathbf{j} and the electric field \mathbf{E} is valid

$$j_\alpha = \sigma_{\alpha\beta}(y)E_\beta, \quad (1)$$

where α, β label the Cartesian components of the magnetoconductivity tensor $\hat{\sigma}$. The inhomogeneity is supposed to be one-dimensional (in y -direction) but both possible geometries of the experiments ($\mathbf{j} \parallel y$ and $\mathbf{j} \perp y$) will be considered. In the geometry of ref. [1] $\mathbf{j} \perp y$ the exact analytical (and quite simple) solution is possible for an arbitrary dependence $\hat{\sigma}(y)$, including the case of nonuniform magnetic field. In the other geometry $\mathbf{j} \parallel y$ the exact analytical (and rather unexpected) result is obtained for the Müller model $B_z = B_0ky$: there exists specific static skin-effect when the current density exponentially depends on the transversal coordinate x .

In both cases I will consider a specimen in the form of a strip of finite width and infinite length. The total current J is fixed (measured by ampermeter) while the electric field components E_x and E_y are to be found from the equations (1) and

$$\text{div} \mathbf{j} = 0, \quad \text{rot} \mathbf{E} = 0. \quad (2)$$

In such a positioning of the problem one does not need to solve the Poisson equation.

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Current perpendicular to the direction of inhomogeneity. The system of equations (1) and () written in the components reads:

$$j_x = \sigma_0 E_x + \sigma_1 E_y, \quad j_y = -\sigma_1 E_x + \sigma_0 E_y, \quad \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} = 0; \quad \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = 0, \quad (3)$$

where $\sigma_0 = \sigma_{xx} = \sigma_{yy}$, $\sigma_1 = \sigma_{xy}$.

Look for the solution with $j_y \equiv 0$. Strictly speaking one needs the solution with $j_y(y=0) = j_y(y=L) = 0$, where L is the width of the strip. However due to the evident unicity of the solution the assumption made does not violate the generality. Then we have:

$$E_y = \frac{\sigma_1}{\sigma_0} E_x = k(y) E_x, \quad j_x = \left(\sigma_0 + \frac{\sigma_1^2}{\sigma_0} \right) E_x = q(y) E_x, \quad k \equiv \frac{\sigma_1}{\sigma_0}, \quad q \equiv \frac{\sigma_0^2 + \sigma_1^2}{\sigma_0} = \frac{1}{\rho_0}, \quad (4)$$

where $\rho_{\alpha\beta}$ is the magnetoresistance tensor, $\rho_0 = \rho_{xx}$. Further:

$$\operatorname{div} \mathbf{j} = \frac{\partial j_x}{\partial x} = q(y) \frac{\partial E_x}{\partial x} = -q(y) \frac{\partial^2 \Phi}{\partial x^2} = 0, \quad (5)$$

with $\mathbf{E} = -\nabla \Phi(x, y)$.

The general solution of eq. (9) has a form

$$\Phi = A(y)x + B(y). \quad (6)$$

It follows from eq. (8)

$$\frac{\partial A}{\partial y} x + \frac{\partial B}{\partial y} = k(y)A(y). \quad (7)$$

Hence,

$$A = \text{const}, \quad B = A \int_{y_0}^y k(y') dy'. \quad (8)$$

The fixed total current determines the value A :

$$J = \int_0^L j_x dy = -A \int_0^L q(y') dy', \quad (9)$$

and the problem is solved. The Hall voltage defined as $\Phi(x, 0) - \Phi(x, L)$ equals

$$V_H = A \int_0^L k(y') dy' = -J \int_0^L k(y') dy' / \int_0^L q(y') dy' \quad (10)$$

and one gets the following expression for the *effective* Hall resistance

$$\rho_1^{\text{eff}} = \langle \frac{\rho_1}{\rho_0} \rangle / \langle \frac{1}{\rho_0} \rangle, \quad (11)$$

where the brackets mean averaging over y :

$$\langle u \rangle \equiv \frac{1}{L} \int_0^L u(y) dy. \quad (12)$$

The diagonal component of $\rho_{\alpha\beta}^{eff}$ is

$$\rho_0^{eff} = \left\langle \frac{1}{\rho_0} \right\rangle^{-1}, \quad (13)$$

that simply corresponds to the parallel connection of conducting filaments stretched along the current direction.

The eqs. (8) and (10) are valid for any kind of a one-dimensional inhomogeneity (e.g. carrier concentration, magnetic field, the density of scatterers). If the specimen is homogeneous and only magnetic field depends on y the relations hold between σ_0 and σ_1 :

$$\frac{\sigma_1}{\sigma_0} = \lambda B(y), \quad \lambda = \text{const}, \quad \frac{\sigma_0^2 + \sigma_1^2}{\sigma_0} \equiv \sigma = \text{const}, \quad (14)$$

following from the classical Drude kinetic theory; σ is the Drude conductivity for $B = 0$. Then $\rho_1^{eff} = \langle \rho_1 \rangle$ that corresponds to the sequential connection of the Hall voltages created by the magnetic field in each conducting filament parallel to Ox . Thus, in the geometry considered in this section, for a homogeneous specimen and inhomogeneous magnetic field the results are quite trivial:

$$\rho_0^{eff} = \left\langle \frac{1}{\rho_0} \right\rangle^{-1}, \quad \rho_1^{eff} = \langle \rho_1 \rangle. \quad (15)$$

Note that for an inhomogeneous specimen, instead of the second relation of eq. (15), a more complicate result (8) is valid.

Current parallel to the magnetic field gradient. Consider now a strip parallel to y and look for the solution with $j_x(y) \equiv 0$. The specimen is supposed to be homogeneous, the local values $\sigma_0(y)$ and $\sigma_1(y)$ are determined by the classical kinetic theory:

$$\sigma_0 = \frac{\sigma}{1 + (\lambda B)^2}, \quad \sigma_1 = \frac{\sigma \lambda B}{1 + (\lambda B)^2}, \quad B = B(y). \quad (16)$$

Then from eqs. (3) with $j_x = 0$ one can easily obtain:

$$j_y = \sigma E_y, \quad \frac{\partial^2 \Phi}{\partial y^2} = 0, \quad E_x = -\lambda B(y) E_y, \quad (17)$$

where the last relation in eq. (12) follows again from $j_x = 0$. Hence,

$$\Phi = C_1(x)y + C_2(x), \quad \frac{\partial C_1}{\partial x} y + \frac{\partial C_2}{\partial x} = -\lambda B(y) C_1(x), \quad (18)$$

and for the Müller model $B(y) = B_0 k y$ the solution has a form

$$C_1 = C e^{-\lambda B_0 k x}, \quad C_2 \equiv 0, \quad \Phi = C y e^{-\lambda k x B_0}. \quad (19)$$

From eq. (19) one obtains

$$E_x = C \lambda B_0 k y e^{-\lambda B_0 k x}, \quad E_y = -C e^{-\lambda B_0 k x}, \quad j_y = -\sigma C e^{-\lambda B_0 k x}, \quad (20)$$

where the constant C can be found via the total current:

$$J = \int_0^L j_y dx = -\frac{\sigma C}{\lambda B_0 k} (1 - e^{-\lambda B_0 k L}). \quad (21)$$

The Hall voltage:

$$\Phi(x=0, y) - \Phi(x=L, y) = y \frac{J \lambda B_0 k}{\sigma}, \quad (22)$$

and the Hall resistance in the point y is

$$\rho_H^{(y)} = \frac{\lambda B_0 k y}{\sigma} = \rho \omega_c(y) \tau, \quad (23)$$

where $\rho = 1/\sigma$, τ is relaxation time and $\omega_c(y)$ is the local value of the cyclotron frequency: $\omega_c(y)\tau = \lambda B(y)$. The most remarkable feature of the obtained solution is the exponential distribution of the current density along the x -direction (see eq. (20)). This can be called the static skin-effect. Depending on the signs of J , B_0 and k the current in y -direction is concentrated either at the left ($x = 0$) or at the right ($x = L$) edge of the strip. The depth of the skin layer l_s is defined by the magnetic field gradient: $l_s = 1/k\omega_{c0}\tau$, where ω_{c0} is the cyclotron frequency for $B = B_0$. The electric field depends also exponentially on the transversal coordinate. Hence, when measuring the Hall voltage V_H between the left edge of the strip $x = 0$ and some variable point x (for the same y !) inside the strip one will find the exponential dependence $V_H(x)$ that would be an experimental evidence of the skin-effect.

Alternate electric field. The above obtained results can be easily extended to the case of finite frequency ω of the electric field if $\omega \ll 1/\tau_M$, where τ_M is the Maxwell relaxation time. For the 3D situation $1/\tau_M = 4\pi\sigma_{3D}$ and for 2D $1/\tau_M = 2\pi\sigma_{2D}/L$. The parameter $\omega\tau$ can be of an arbitrary magnitude. By making use of the well known formulae for $\sigma_{\alpha\beta}(B)$ allowing for the dispersion one can easily see that it is necessary just to substitute

$$\sigma \rightarrow \frac{\sigma}{1 - i\omega\tau}$$

in all the preceding formulae. For example, the Hall voltage between the points $(0, y)$ and (x, y) reads

$$V_H(0, x; y) = CyR_e \left\{ \left[1 - \exp \frac{(-kx)\omega_{c0}\tau(1 + i\omega\tau)}{1 + \omega^2\tau^2} \right] e^{-i\omega t} \right\}. \quad (24)$$

Thus the voltage and the current density decay with oscillations when the distance from the strip edge increases.

How to realize the linearly nonuniform magnetic field. Here I consider only the case of a 2D system. As 2D electrons "feel" only the normal component of the magnetic field B_n the inhomogeneity of B_n can be achieved for a thin conducting film bent to a proper shape and placed in a uniform field $\mathbf{B}(0, 0, B)$. The dependence $B_n(y) = B_0ky$ is realized for the cylindrical surface $z = F(y)$, where

$$F(y) = \pm \frac{1}{2k} \left[\sqrt{2ky(1 - 2ky)} + \arcsin \sqrt{2ky} \right], \quad 0 \leq ky \leq \frac{1}{2}. \quad (25)$$

In conclusion, the exact analytical solutions are obtained for the classical Hall effect in inhomogeneous systems. If the current flows perpendicularly to the inhomogeneity the solution is possible in the quite general form. For the parallel orientation of the current and magnetic field gradient the analytical solution is found for the linearly inhomogeneous magnetic field. In the latter case the static skin-effect occurs for a specimen in the form of a long strip.

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1. J.E.Müller, Phys. Rev. Lett. **68**, 385(1992).