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OVERLAP INTEGRAL FOR QUANTUM SKYRMIONS

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Making use of the method of spin coherent states we have obtained an analytical form for overlap integral for quantum skyrmions.

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Skyrmions are general static solutions of 2D Heisenberg ferromagnet, obtained by Belavin and Polyakov [1] from classical nonlinear sigma model. A renewed interest to these unconventional spin textures is stimulated by high-T_c problem in doped quasi-2D cuprates and quantum Hall effect.

The spin distribution within classical skyrmion of topological charge $q = 1$ is given as follows

$$S_x = \frac{2r\lambda}{r^2+\lambda^2}\cos\varphi, \qquad S_y = \frac{2r\lambda}{r^2+\lambda^2}\sin\varphi, \qquad S_z = \frac{r^2-\lambda^2}{r^2+\lambda^2}. \tag{1}$$

In terms of the stereographic variables the skyrmion with radius λ and phase φ_0 centered at a point z_0 is identified with spin distribution $w(z) = \Lambda/(z - z_0)$, where $z = x + iy = re^{i\varphi}$ is a point in the complex plane, $\Lambda = \lambda e^{i\theta}$, and characterised by three modes: translational z_0-mode, "rotational" θ-mode and "dilatational" λ-mode. Each of them relates to certain symmetry of the classical skyrmion configuration. For instance, θ-mode corresponds to combination of rotational symmetry and internal $U(1)$ transformation.

The simplest wave function of the spin system, which corresponds to classical skyrmion, is a product of spin coherent states [2]. In case of spin $s = \frac{1}{2}$

$$\Psi_{sk}(0) = \prod_i [\cos\frac{\theta_i}{2}\exp\left\{i\frac{\varphi_i}{2}\right\}|\uparrow\rangle + \sin\frac{\theta_i}{2}\exp\left\{-i\frac{\varphi_i}{2}\right\}|\downarrow\rangle], \tag{2}$$

where $\theta_i = \arccos(r_i^2 - \lambda^2)/(r_i^2 + \lambda^2)$. Coherent state implies a maximal equivalence to classical state with minimal uncertainty of spin components. In this connection we should note that such a state was used in paper [3] by Schliemann and Mertens where an expression for the square variance of the Heisenberg Hamiltonian was obtained.

Classical skyrmions with different phases and radia have equal energy. Nevertheless, stationary state of quantum skyrmion of topological charge $q = 1$ is not a superposition of states with different phase and radius [4], but has a certain distinct value of λ.

In this paper [5] we consider some peculiarities of quasiparticle behaviour for the quantum skyrmion. First of all it implies the calculation of the overlap and "resonance" integrals: $S_{12} = \langle\Psi_{sk}(\mathbf{R}_1)|\Psi_{sk}(\mathbf{R}_2)\rangle$, $H_{12} = \langle\Psi_{sk}(\mathbf{R}_1)|\hat{H}|\Psi_{sk}(\mathbf{R}_2)\rangle$.

As a helpful illustration to the calculation of the overlap integral for quantum topological defects it is worth to present some known results concerning the overlap of vortices in

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2D superconducting condensate [6]. Phenomenologically such vortices were described as point quasiparticles, moving under the action of the transverse Magnus force. Coherent state of the vortex with the center in $\mathbf{R}_0$ was taken in the form [6]

$$|\Psi_{sk}(\mathbf{R}_0)\rangle = \frac{1}{\sqrt{2\pi l^2}} \exp\left[-\frac{|\mathbf{r}-\mathbf{R}_0|^2}{4l^2} + \frac{i\mathbf{z}\cdot\mathbf{R}_0\times\mathbf{r}}{2l^2}\right], \tag{3}$$

where ρ_0 is a density of 2D condensate, $l = (2\pi\rho_0)^{-1/2}$ is an average distance between particles in the condensate, $\mathbf{z}$ a unit vector normal to the plane of condensate. Overlap integral for two coherent states is then easily calculated as

$$S_{12} = \langle\Psi_{sk}(\mathbf{R}_1)|\Psi_{sk}(\mathbf{R}_2)\rangle = \exp[-\frac{1}{4l^2}(R_{12}^2 - 4i\Delta_{12})], \tag{4}$$

where

$$\Delta_{12} = \frac{1}{2}\mathbf{z}\cdot[\mathbf{R}_1\times\mathbf{R}_2], \quad R_{12} = |\mathbf{R}_1 - \mathbf{R}_2|.$$

It contains Gaussian factor reflecting localisation of the coherent state and also a specific phase factor Δ_{12}, being the area of a sector topological defect covers while moving in the plane. This factor originates from the phase factor in the function (3) typical for the charged particle moving in the magnetic field or, in general, for particle which experiences a Magnus force. The correct form of such a Magnus force for the out-of-plane magnetic vortex with topological charge $\frac{1}{2}$ ("half-skyrmion") was derived by Nikiforov and Sonin [7], the Magnus force acting on the classical skyrmion is simply twice larger [8].

It seems, the expression (4) reflects two common features of the overlap integral for topological defects in 2D system, namely Gaussian dependence on R_{12} and presence of the Berry phase. One should note that in recent paper by Thang [9] it is shown that wave function (3) does not provide correct description of the transition to the infinite system, when the overlap integral turns to zero. Nevertheless, the expressions (3) and (4) for the wave function and overlap integral allow to elucidate many generic features of the corresponding quantities for the topological defects.

To reveal some peculiarities of the quantum quasiparticle behaviour for skyrmions we consider overlap integral for the simple quantum state like (2) of spin system with skyrmion $\lambda/(z-R_1)$ at the point $R_1 = \mid R_1 \mid e^{i\varphi_1}$ with the state $\lambda/(z-R_2)$, which corresponds to skyrmion shifted to an arbitrary distance

$$R_1^2 = \mid R_1 \mid^2 + \mid R_2 \mid^2 - 2R_1R_2\cos(\varphi_1-\varphi_2)$$

into the point $R_2 = \mid R_2 \mid e^{i\varphi_2}$. Overlap of the single spin coherent states characterised by two different points on the complex plane ς and μ, is [2]

$$\langle\varsigma|\mu\rangle = \frac{(1+\varsigma\overline{\mu})^{2S}}{(1+\mid\varsigma\mid^2)^S(1+\mid\mu\mid^2)^S}. \tag{5}$$

Thus, overlap integral for the skyrmion states is given by

$$S_{12}^{\lambda\lambda} = \prod_i \frac{(1+\varsigma\overline{\mu})^{2S}}{(1+\mid\varsigma\mid^2)^S(1+\mid\mu\mid^2)^S} = \exp(S\sum_i \ln[\frac{(1+\varsigma\overline{\mu})^2}{(1+\mid\varsigma\mid^2)(1+\mid\mu\mid^2)}]) =$$

$$= \exp(S\sum_i[2\ln(1+\frac{\lambda^2}{r_i^2+R_1\overline{R_2}-r_iR_1e^{-i\varphi_i}-r_i\overline{R_2}e^{+i\varphi_i}} -$$

$$-\ln(1+\frac{\lambda^2}{r_i^2+R_2^2-2R_2r_i\cos\varphi_i})-\ln(1+\frac{\lambda^2}{r_i^2+R_1^2-2R_1r_i\cos\varphi_i})]])=$$

$$=\exp(2I(R_1,R_2)-I(R_1,R_1)-I(R_2,R_2). \tag{6}$$

It should be noted that for displacements $R \geq 2\lambda$ on continuous complex plane the overlap integral turns to zero since in this case there exist two such points for which initial and final spin states are orthogonal. For example, in case of $R_1 = 0, R_2 = R$, where R is real, $1 + \lambda^2/z(z - R) = 0$, when $z = R/2 \pm \sqrt{R^2/4 - \lambda^2}$. However, overlap of the skyrmion on the lattice turns to zero only at certain values of R, which form a certain discrete set of values. In particular, for displacement along the direction of elementary vector of the lattice overlap is identically zero at the points $R = (\lambda^2 + k^2)/k$ given integer k.

To consider the skyrmions of large radius we turn the sum in the exponent of Eq. (6) into integral. The quantity c/λ for large skyrmions is small and virtually we need to keep terms of zero order in c/λ, with c being lattice constant. Spin density, that is the number of spins in the unit cell of the plane, is simply $1/c^2$. Theory will be invariant with respect to scale transformations, that is to variations of c and simultaneous equivalent variation of λ and R. Besides, we consider size L of the system to be much larger than skyrmion size and virtually keep only terms of zero order in λ/L. To this end we may simply set $L = \infty$. So, the result is

$$I(R_1,R_2)=\frac{1}{c^2}\int_0^\infty A(r,R_1,R_2)\,r\,dr, \tag{7}$$

where

$$A(r,R_1,R_2)=\int_0^{2\pi} d\varphi\,\ln(\lambda^2+r^2+R_1\overline{R_2}-rR_1e^{-i\varphi}-r\overline{R_2}e^{+i\varphi}). \tag{8}$$

In order to perform the angular integration in $A(r, R_1, R_2)$, we introduce complex variable $z = e^{+i\varphi}$, so that

$$A(r,R_1,R_2)=\int dz\frac{ln(\lambda^2+r^2+R_1\overline{R_2}-rR_1z^{-1}-r\overline{R_2}z)}{iz}, \tag{9}$$

where the integration is taken on the circle of unit radius. This integral appears to be nonzero due to the pole in $z = 0$ and the nonanaliticity area connected with the existence of the cut in the space of values of the complex logarithm when its argument is negative. While crossing the axis of negative values of the argument the phase jumps from $-\pi$ to π. In the plane of z nonanaliticity area is the curve given by the equation

$$\mathrm{Im}(\lambda^2+r^2+R_1\overline{R_2}-rR_1z^{-1}-r\overline{R_2}z)=0,$$

$$\mathrm{Re}(\lambda^2+r^2+R_1\overline{R_2}-rR_1z^{-1}-r\overline{R_2}z)<0.$$

When $R > 2\lambda$ there exist interval of the values r, for which the end of the cut lies on the border of the unit circle resulting in purely imaginary contribution into integral $A(r, R_1, R_2)$. The range of r, in which this occurs is given by the interval $[r_1, r_2]$, where

$$r_{1,2}^2=(\mid R_1\mid^2+\mid R_1\mid^2)/2-\lambda^2\pm(\mid R_1\mid^2-\mid R_1\mid^2)\sqrt{\frac{1}{4}-\frac{\lambda^2}{R^2}}.$$

As a result of existence of the cut we have the following expression for the integral

$$A(r, R_1, R_2) = 2\pi \begin{cases} \ln(\frac{\lambda^2+r^2+R_1\overline{R_2}+\sqrt{(\lambda^2+r^2+R_1\overline{R_2})^2-4r^2R_1\overline{R_2}}}{2}), & \text{if } r > r_2 \text{ or } r < r_1 \\ \ln(rR_1) - i \arcsin y(r), & \text{if } r_1 < r < r_2 \end{cases},$$

where

$$y(r) = \mid R_2 \mid \sin f \frac{- \mid R_1 \mid^2 + \mid R_1 \mid\mid R_2 \mid \cos f + \sqrt{r^2R^2 - \mid R_1 \mid^2 \mid R_2 \mid^2 \sin^2 f}}{rR^2},$$

with f being the phase difference between $R_1, \overline{R_2}$. Similar expressions exist for $A(r, R_1, R_1)$ and $A(r, R_2, R_2)$. With account of obvious identity

$$((\lambda^2 + r_1^2 + R_1\overline{R_2})^2 - 4r_1^2 R_1\overline{R_2})-$$

$$-((\lambda^2 + r_2^2 + R_1\overline{R_2})^2 - 4r_2^2 R_1\overline{R_2}) =$$

$$= (R^2 + 2i \sin f \mid R_1 \mid\mid R_2 \mid)\sqrt{\frac{1}{4} - \frac{\lambda^2}{R^2}}(\mid R_1 \mid^2 - \mid R_2 \mid^2))$$

and integration in (7) we obtain finally for the overlap integral

$$S_{12} = \exp[-\frac{\pi S}{c^2}(R_{12}^2 - 4i\Delta_{12})], \tag{10}$$

if $R_{12} < 2\lambda$ or

$$S_{12} = \exp[-\frac{S\pi}{c^2}(R_{12}^2 - 4i\Delta_{12} - R_{12}\sqrt{R_{12}^2 - 4\lambda^2} + 2\lambda^2 \ln(\frac{R_{12} + \sqrt{R_{12}^2 - 4\lambda^2}}{R_{12} - \sqrt{R_{12}^2 - 4\lambda^2}})], \tag{11}$$

if $R_{12} > 2\lambda$. As above in expression (3) here Δ_{12} is an area of a sector skyrmion covers while moving in the plane. Skyrmion, having wandered closed contour on the plane, acquires the phase $4\pi S(\Delta_{12}/c^2)$, that is the skyrmion accumulates a phase of $4\pi S$ for every spin it encircles. So, its quantum motion looks like that of charged particle with unit charge in a uniform magnetic field of strength $4\pi\hbar S/c^2$ [10], or particle which experiences a transverse Magnus force $[\mathbf{b} \times \mathbf{v}]$, where $\mathbf{b} = 4\pi\hbar S\mathbf{z}/c^2$, $\mathbf{v}$ - velocity, so that corresponding length scale is $(4\pi S/c^2)^{-1/2}$ which is similar to magnetic length. This is the same Magnus force that acts on the classical skyrmion [8] and which is simply twice larger than one acting on the out-of-plane magnetic vortex with topological charge $\frac{1}{2}$ ("half-skyrmion") [7].

One should note the specific dependence of the overlap integral on the spin-site density $(1/c^2)$ and the number of spin deviations $(2S)$ similar to the density dependence in expression (3).

When $R \to \infty$ the R-dependence of the overlap integral obeys a power low

$$S_{12} = (\frac{\lambda^2}{R^2})^{S2\pi\lambda^2/c^2} e^{-S2\pi\lambda^2/c^2}.$$

The expressions for the overlap of skyrmions were obtained in continuous approximation which generally is not correct in the vicinity of those values of displacements $R_{12} > 2\lambda$, at which the overlap of distinct spin states is zero. Nonetheless, direct numerical calculation for large enough 2D lattices shows that logarithm of the skyrmion overlap integral is "almost everywhere" satisfactorily described by the continuous expression (11) except for discrete set of values of R_{12}, in which logarithm turns to $-\infty$.

We considered everywhere the overlap of the skyrmions with equal λ, that is with equal global phase and radius. It is easy to see, that skyrmion states with different λ and phases were orthogonal: $S_{12}^{\lambda\varphi\lambda'\varphi'} \propto \delta_{\lambda\lambda'}\delta_{\varphi\varphi'}$, which agrees with the "conservation low" of the quantity λ and phase in the skyrmion [4].

In conclusion, we made use a simplified form for the quantum skyrmion wave function to obtain the analytical expression for appropriate overlap integral, which form confirms a known statement [7, 8, 10] that skyrmion moves like a particle in uniform magnetic field, or particle which experiences a Magnus force. We hope the making use of more realistic wave function will keep up principal features of above derived result.

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