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SEMICLASSICAL LIMIT FOR CHERN-SIMONS THEORY ON COMPACT HYPERBOLIC SPACES

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The invariant integration method for Chern-Simons theory defined on compact hyperbolic spaces of the form $\Gamma\backslash\mathbb{H}^3$ is verified in the semiclassical approximation. The semiclassical limit for the partition function is calculated. We discuss briefly a contribution to the sum over topologies in 3-dimensional quantum gravity.

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Introduction. It is known that topological invariants associated with 3-manifolds can be constructed within the framework of Chern-Simons gauge theory [1]. These values were specified in terms of the axioms of topological quantum field theory [2], whereas equivalent derivation of invariants was also given combinatorially in [3, 4], where modular Hopf algebras related to quantum groups have been used. The Witten's (topological) invariants have been explicitly calculated for a number of 3-manifolds and gauge groups [5 – 11]. The semiclassical approximation for the Chern – Simons partition function $\mathfrak{W}(k)$ can be given by the asymptotic $k \to \infty$ of Witten's invariant of a 3-manifold M and a gauge group G. Typically this expression is a partition function of quadratic functional.

This note is an extension of the previous paper [12]. Our aim here will be to use the invariant integration method [13, 14] in its simplest form for the semiclassical approximation for Chern – Simons theory, defined on hyperbolic 3-manifolds of the form $M = \Gamma\backslash\mathbb{H}^3$, where $\mathbb{H}^3$ is the Lobachevsky space and Γ is a co-compact discrete group of isometries (see Ref. [15] for detail).

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The semiclassical approximation for the partition function. The partition function associated to Chern – Simons gauge theory has the form

$$\mathfrak{W}(k) = \int \mathcal{D}A \exp[ikCS(A)], \qquad k \in \mathbb{Z}, \tag{1}$$

where

$$CS(A) = \frac{1}{4\pi}\int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right). \tag{2}$$

The quantity $\mathfrak{W}(k)$ is a (well-defined) topological invariant of M. The formal integration in (1) is over the gauge fields A in a trivial bundle, i.e. 1-forms on a 3-dimensional manifold M with values in Lie algebra $\mathbf{g}$ of gauge group G.

In the limit $k \to \infty$ Eq. (1) is given by its semiclassical approximation, involving only partition functions of quadratic functionals [1]:

$$\sum_{[A_f]} \exp[ikCS(A_f)] \int \mathcal{D}B \exp\left(\frac{ik}{4\pi}\int_M \mathrm{Tr}\left(B \wedge d_{A_f}B\right)\right). \tag{3}$$

In Eq. (3) the sum is taken over representatives A_f for each point $[A_f]$ in the moduli-space of flat gauge fields on M. In addition, B is Lie-algebra-valued 1-form and d_{A_f} is the covariant derivative determined by A_f, namely $d_{A_f}B = dB + [A_f, B]$.

We shall use the invariant integration method [13, 14], which enables the partition functions in Eq.(3) to be evaluated in complete generality. Let M be a compact oriented Riemannian manifold without boundary, and $n = 2m+1 = \dim M$ is the dimension of the manifold. Let $\chi : \pi_1(M) \longmapsto O(V, \langle\cdot,\cdot\rangle_V)$ be a representation of $\pi_1(M)$ on real vectorspace V. The mapping χ determines (on a basis of standard construction in differential geometry) a real flat vectorbundle ξ over M and a flat connection map ∇_p on the space $\Omega^p(M,\xi)$ of differential p-forms on M with values in ξ. One can say that χ determines the space of smooth sections in the vectorbundle $\Lambda^p(TM)^* \otimes \xi$. One can construct from the metric on M and Hermitian structure in ξ a Hermitian structure in $\Lambda(TM)^* \otimes \xi$ and the inner products $\langle\cdot,\cdot\rangle_m$ in the space $\Omega^m(M,\xi)$. Thus

$$S_{\mathcal{O}} = \langle \omega, \mathcal{O}\omega\rangle_m, \qquad \mathcal{O} = *\nabla_m, \tag{4}$$

where $(*)$ is the Hodge-star map. The map $\mathcal{O}$ is formally self-adjoint with the property $\mathcal{O}^2 = \nabla_m^*\nabla_m$. Suppose that the quadratic functional (4) is defined on the space $\mathcal{G} = \mathcal{G}(M,\xi)$ of smooth sections in a real Hermitian vectorbundle ξ over M. There exists a canonical topological elliptic resolvent $R(S_{\mathcal{O}})$, related to the functional (4), namely

$$0 \xrightarrow{0} \Omega^0(M,\xi) \xrightarrow{\nabla_0} \dots \xrightarrow{\nabla_{m-2}} \Omega^{m-1}(M,\xi) \xrightarrow{\nabla_{m-1}} \ker(S_{\mathcal{O}}) \xrightarrow{0} 0. \tag{5}$$

Therefore, for the resolvent $R(S_{\mathcal{O}})$, we have $\mathcal{G}_p = \Omega^{m-p}(M,\xi)$ and $H^p(R(S_{\mathcal{O}})) = H^{m-p}(\nabla)$, where $H^p(\nabla) = \ker(\nabla_p)/\mathrm{Im}(\nabla_{p-1})$ are the cohomology space. Note that $S_{\mathcal{O}} \geq 0$ and therefore $\ker(S_{\mathcal{O}}) \equiv \ker(\mathcal{O}) = \ker(\nabla_m)$. Let us choose an inner product $\langle\cdot,\cdot\rangle_{H^p}$ in each space $H^p(R(S_{\mathcal{O}}))$.

The partition function of $S_{\mathcal{O}}$ with resolvent (5) can be written in the form [14, 16]:

$$\mathfrak{W}(k) \equiv \mathfrak{W}(k; R(S_{\mathcal{O}}), \langle\cdot,\cdot\rangle_H, \langle\cdot,\cdot\rangle) = (\pi/k)^{\zeta(0,|\mathcal{O}|)/2} e^{-\frac{i\pi}{4}\eta(0,\mathcal{O})} \times$$

$$\times\tau(M,\chi,\langle\cdot,\cdot\rangle_H)^{1/2}, \tag{6}$$

where $|\mathcal{O}| = \sqrt{\mathcal{O}^2}$ is defined via spectral theory. This is the basic formula one has to evaluate.

It can be shown that the zeta function $\zeta(s,|\mathcal{O}|)$ appearing in the partition function (6) is well-defined and analytic for $\mathrm{Re}\, s > 0$ and can be continued to a meromorphic function on $\mathbb{C}$, regular at $s = 0$. Furthermore, the zeta function can be expressed in terms of the dimensions of the cohomology spaces of $\mathcal{O}$. Since $H^p(R(S_{\mathcal{O}})) = H^{m-p}(\nabla)$ (the Poincarè duality) for the resolvent (5), it follows that (see Refs. [14, 16] for details)

$$\zeta(0||\mathcal{O}|) = -\sum_{p=0}^{m}(-1)^p \mathrm{dim}H^p(R(S)) = (-1)^{m+1}\sum_{p=0}^{m}(-1)^p\mathrm{dim}H^p(\nabla). \tag{7}$$

The dependence of the eta invariant $\eta(0|\mathcal{O})$ of Atiyah – Patodi – Singer [17 – 19] on the connection map $\mathcal{O}$ can be expressed with the help of the formula for the index of the twisted signature operator for a certain vectorbundle over $M \otimes [0,1]$. Furthermore it can be shown [17] that $\eta(s|B) = 2\eta(s|\mathcal{O})$, where the B are elliptic self-adjoint maps on $\Omega(M,\xi)$ defined on p-forms by

$$B_p = (-i)^{\lambda(p)}\left(*\nabla + (-1)^{p+1}\nabla *\right). \tag{8}$$

In this formula $\lambda(p) = (p+1)(p+2)+m+1$ and for the Hodge star-map we have used $*\alpha\wedge\beta = \langle\alpha,\beta\rangle_{vol}$. From the Hodge theory we have

$$\mathrm{dimker}B = \sum_{p=0}^{m}\mathrm{dim}H^p(\nabla).$$

Finally the quantity $\tau(M,\chi,\langle\cdot,\cdot\rangle_H)$ is related to the Ray – Singer (analytic) torsion $T_{an}^{(2)}(M)$. In fact, if $H^0(\nabla)\neq 0$ and $H^p(\nabla) = 0$ for $p = 1,...,m$, then the product

$$\tau(M,\chi,\langle\cdot,\cdot\rangle_H) = T_{an}^{(2)}(M)\cdot \mathrm{Vol}(M)^{-\mathrm{dim}H^0(\nabla)}, \tag{9}$$

is metric independent [20], i.e. the metric dependence of the Ray – Singer torsion factors out as $V(M)^{-\mathrm{dim}H^0(\nabla)}$.

The case of real compact hyperbolic manifolds. Let us consider the specific case of a compact hyperbolic 3-manifolds of the form $M = \Gamma\backslash\mathbb{H}^3$. If the flat bundle, ξ is acyclic, then for analytic torsion one gets [21]:

$$[T_{an}^{(2)}(X_\Gamma)]^2 \equiv \mathfrak{R}_\chi(0) = \prod_{p=0}^{\mathrm{dim}M}[\mathrm{det}\Delta_p]^{(-1)^{p+1}p/2}, \tag{10}$$

where $\mathfrak{R}_\chi(s)$ is the Ruelle function and Δ_p is the Laplacian restricted on p-forms and the determinants are defined by means of zeta-regularization. The function $\mathfrak{R}_\chi(s)$ extends meromorphically to the entire complex plane $\mathbb{C}$; it is an alternating product of more complicate factors, each of which is a Selberg zeta function $Z_p(s;\chi)$. The Ruelle function associated with closed oriented hyperbolic 3-manifold $\Gamma\backslash\mathbb{H}^3$ has the form $\mathfrak{R}_\chi(s) = Z_0(s;\chi)Z_2(2+s;\chi)/Z_1(1+s;\chi)$. For the Ray – Singer torsion one gets [12]

$$[T_{an}(\Gamma\backslash H^3)]^2 = \mathcal{R}_\chi(0) = \frac{[Z_0(2,\chi)]^2}{Z_1(1,\chi)}\exp\left(-\frac{\mathrm{Vol}(\Gamma\backslash\mathbb{H}^3)}{3\pi}\right), \tag{11}$$

where $\mathrm{Vol}(\Gamma\backslash\mathbb{H}^3)$ is the volume of a fundamental domain of $\Gamma\backslash\mathbb{H}^3$. In the presence of non-vanishing Betti numbers $b_j = b_j(\Gamma\backslash\mathbb{H}^3)$ we have [12, 22]

$$[T_{an}(\Gamma\backslash H^3)]^2 = \frac{(b_1-b_0)![Z_0^{(b_0)}(2,\chi)]^2}{[b_0!]^2 Z_1^{(b_1-b_0)}(1,\chi)} \exp\left(-\frac{\mathrm{Vol}(\Gamma\backslash\mathbb{H}^3)}{3\pi}\right). \tag{12}$$

Now we consider the contribution associated with eta invariant. A remarkable formula relating $\eta(s,\mathcal{O})$ to the closed geodesics on $\Gamma\backslash\mathbb{H}^3$ has been obtained by Millson [23]. More explicitly, Millson has proved the following result for a Selberg type (Shintani) zeta function $\widetilde{Z}(s,\mathcal{O})$, which admits a meromorphic continuation to the entire complex plane. $\widetilde{Z}(s,\mathcal{O})$ is a holomorphic function at $s=0$ and

$$\log\widetilde{Z}(0,\mathcal{O}) = \pi i\eta(0,\mathcal{O}). \tag{13}$$

Furthermore, it is possible to show that $\widetilde{Z}(s,\mathcal{O})$ satisfies the functional equation $\widetilde{Z}(s,\mathcal{O})\widetilde{Z}(-s,\mathcal{O}) = e^{2\pi i\eta(0,\mathcal{O})}$.

Now we have all the ingredients for the evaluation of the partition function (6) in terms of Ray – Singer torsion and a Selberg type function. The final result is

$$\mathfrak{W}(k) = \left(\frac{\pi}{k}\right)^{\zeta(0,|\mathcal{O}|)/2}\left[\frac{\mathcal{R}_\chi(0)}{\widetilde{Z}(0,\mathcal{O})}\right]^{1/4}[\mathrm{Vol}(\Gamma\backslash G)]^{-\mathrm{dim}H^0(\nabla)/2}, \tag{14}$$

where $\zeta(0,\mathcal{O})$ is given by Eq. (7).

Concluding remarks. We have derived the explicit formula for the semiclassical approximation for the Chern – Simons partition function, using the invariant integration method. The final formula are given in a form where the behaviour as $k\mapsto\infty$ is obvious. In this connection we have explicitly exhibited the first term in the level k asymptotic expansion for compact hyperbolic families of 3-manifolds.

The evaluation of the Ray – Singer torsion presented in this paper may be useful within the Euclidean path-integral approach to 3-dimensional quantum gravity, where the partition function is evaluated by summing contributions from all possible topologies [24]. For negative cosmological constant Λ, the classical extrema of the Euclidean action are hyperbolic manifolds. It has been shown that 3-dimensional gravity can be rewritten as a Chern – Simons theory for a suitable gauge group [25]. Therefore in the one-loop partition function the quantum prefactor turns out to be dependent on the Ray – Singer torsion of a hyperbolic manifold. Note that the dependence on the volume of the Ray – Singer torsion is exponentially decreasing, making a contribution to the one-loop Euclidean partition function of the same nature of the one corresponding to the classical action. Namely, the one-loop Euclidean partition function, including only one extremum with $\Lambda<0$ and in absence of zero modes, reads (see also Ref. [12])

$$\mathfrak{W}_{\Gamma\backslash\mathbb{H}^3} = \left[\frac{\mathcal{R}_\chi(0)}{\widetilde{Z}(0,\mathcal{O})}\right]^{1/4}\exp\left[-\frac{\mathrm{Vol}(\Gamma\backslash\mathbb{H}^3)}{4\pi G}\left(\frac{1}{G|\Lambda|^{1/2}}+\frac{1}{3}\right)\right], \tag{15}$$

where the second term in the exponential is the first quantum correction.

Note that there is a class of compact sufficiently large hyperbolic manifolds which admit arbitrarly large values of $b_1(M)$. In general, hyperbolic manifolds have not been completely classified and therefore a systematic computation is not yet possible. However this is not the case for certain sufficiently large manifolds, the Haken manifolds [26]. There exists an algorithm for the enumeration of all Haken manifolds and there exists an algorithm for recognizing homeomorphy of the Haken manifolds [27]. These manifolds give an essentional contribution to the partition functions (14) and (15).

Finally, the explicit result (14) can be very important for investigating the relation between quantum invariants for an oriented 3-manifold, defined with the help of a representation theory of quantum groups [3, 4], and Witten's invariant [1], which is, instead, related to the path integral approach.

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