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# ON A BANKS – CASHER RELATION IN MQCD

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We discuss the meaning of a Banks – Casher relation for the Dirac operator eigenvalues in MQCD. It is argued that the eigenvalue can be identified with a coordinate involved in the string compactification manifold.

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**1.** Brane approach to the SUSY gauge theories in different dimensions seems to be the most promising tool to capture their nonperturbative features. It is believed that all universal characteristics of the vacuum sector can be seen in terms of brane picture. Recently a proper brane configuration was found in IIA [1] and $M$ theory [2, 3] for the theories with $N = 1$ supersymmetry. Configuration for the pure gauge theory suggested in [1] involves NS5 and D4 branes which being lifted to $M$ theory are identified with the single M5 brane wrapped around Riemann surface embedded into the three dimensional complex space.

It is known that gluino condensate is developed in $N = 1$ SQCD and properties of the vacuum sector are governed by superpotentials generated in different ways depending on the relation between $N_f$ and $N_c$. Gluino condensate indicates the chiral symmetry breaking so its derivation in MQCD framework is very important. It was found that superpotentials can be calculated in $M$ theory [3, 2, 4] moreover it turns out that superpotential can be attributed to the 5 brane instantons [5, 6]. General consideration of the chirality along the brane approach indicates that at least in some situations it depends only on the local properties of the brane configuration [7]. It was also found [8] that algebra of $N = 1$ SUSY implies the existence of the domain wall between the space regions with different phases of the gluino condensate.

In this note we consider the Banks – Casher relation [9] relating the properties of the Dirac operator eigenvalues in the instanton ensemble background with the value of the vacuum fermion condensate. This relation has been known in QCD for a while where fermions in the fundamental representation condense in the vacuum. Generalization to the adjoint fermions relevant for the supersymmetric theory as well as additional sum rules for the spectral density of the Dirac operator have been discovered in [10]. Spectral density is the universal characteristics of the low energy sector of the theory so one expects that it can be also treated in the brane terms. Note that universal behaviour of the spectral density admits matrix model approach for the investigation of its properties (see [11] for a review) which sucsessfully describes key features of the spectrum. Below we suggest interpretation of a Banks – Casher relation in terms of M5 brane worldvolume and discuss the possible role of the Dirac operator eigenvalue as one of the dimensions involved in the brane picture.

**2.** Let us remind the main facts about the spectrum of Dirac operator in QCD. One concerns eigenvalues of the operator $\hat{D}\Psi = \lambda\Psi$ and density of the eigenvalues $\rho(\lambda)$ provides the order parameter in the low energy sector. Namely due to Casher – Banks relation [9] for the fundamental fermions

$$< \bar{\Psi}\Psi >= -\pi\rho(0) = -\int \frac{\rho(\lambda)d\lambda}{\lambda}, \tag{1}$$

density at origin can be considered as the derivative of the partition function with respect to the mass $\pi\rho(0) = -dlnZ/dm$ at $m = 0$. More generally, there is an infinite tower of sum rules for the inverse powers of eigenvalues [10], for instance

$$< \sum \frac{1}{\lambda_i^2} >_\nu = \frac{(< \bar{\Psi}\Psi > V)^2}{4(\mid \nu \mid + N_f)}, \tag{2}$$

where the averaging with the spectral density is implied, $V$ is the four dimensional Euclidean volume and $\nu$ is the topological charge of the gauge field configuration. Sum rules probe the structure of the small $\lambda$ region of the spectrum.

Another approach to the same object comes from the representation of the partition function as the path integral averaged fermion determinant

$$Z(m) =< Det(iD - m) > . \tag{3}$$

Averaging can be substituted by the integration over the instanton moduli space which due to ADHM description is modelled by the proper matrix model [11]. Along this way of reasoning it is possible to derive the eigenvalue distribution itself. Actually determinant can be considered as the observable in the theory dealing with the instanton moduli space. It is generally believed that the spectrum in the QCD instanton vacuum acquires the band structure due to the delocalization of the zero mode on the single instanton in the instanton ensemble. It is assumed that zero eigenvalue lies in the allowed band providing the chiral symmetry breaking.

A little bit different way to capture the fermion condensate is to consider the analytic properties of the resolvent of the Dirac operator considered on the complex mass plane

$$G(z) =< \mathrm{Tr}\frac{1}{iD - z} > . \tag{4}$$

Riemann surface of $G(z)$ has the cut along the imaginary axis which is in one-to-one correspondence with the presence of the condensate.

**3.** Let us now proceed to the supersymmetric case. To start note that now we consider the condensate of gluinos-fermions in the adjoint representation. Banks-Casher relation for the adjoint fermions holds true [10] so the issue of the derivation of the gluino condensate via the spectral density is well defined.

We would like to obtain the brane interpretation of the Banks – Casher relation as well as some features of the spectral density. $M$ theory picture amounts to the $N=2$ SUSY gauge theory on the world-wolume of the $M$ theory fivebrane wrapped around the surface $\Sigma$ [12] (see also [13])

$$t^2 - P_n(v)t + 1 = 0. \tag{5}$$

In what follows we would like to present the arguments that $v$ can be identified with the eigenvalue of the 4d Dirac operator. To treat the $N=1$ theory we should proceed in two steps. First, we have to shrink all monopole cycles on the surface $\Sigma$ reducing the curve to

$$y^2 = (\frac{v^2}{4} - 1)Q_{n-1}^2(v); \quad y = t - t^{-1}, \tag{6}$$

where $Q_n$ are the Chebyshev polynomials. Then one performs "rotation" which embedds the new curve into three dimensional complex space. The resulting rational curve has the form

$$v = t^n; \quad vw = \theta, \tag{7}$$

where $\theta$ can be identified with the gluino condensate in [2, 3]. We would like to consider the equation

$$< \xi\xi > = \int w dv, \tag{8}$$

where $\xi$ is the gluino field, as Banks - Casher relation. Since in the IIA language D4 branes are located between two NS5 branes this identification implies that the condensate develops purely due to the presense of one NS5 brane at $v=0$.

Let us comment on the interpretation of $v$,$w$ and $t$ variables in the field theory framework. In the theory with $N=4$ supersymmetry it is straightforward to identify six dimensions with the zero modes of three complex scalar fields. If one treats $N=2$ theories starting from softly broken $N=4$ ones then the trace from two complex dimensions corresponding to the massive scalar fields survives due to the dimensional transmutation. We conjectured quite different interpretation in $N=1$ theory assuming that $v$ is the eigenvalue of the Dirac operator. Therefore coordinates $w$ and $t$ have to be considered as the eigenvalues of operators commuting with the Dirac operator and therefore can be associated with the global symmetries.

Let us compare our interpretation of $N=1$ theory with $N=2$ one and show that there is qualitative agreement with the standard treatment of these theories. We will use arguments coming from approach based on the relation of $N=2$ theories to the integrable systems [14].

In IIA picture $N=2$ theory lives on the worldvolume of $N_c$ D4 branes with finite extent in $x_6$ dimension [12]. Parallel NS5 branes are located at $x_6 = 0$ and $x_6 = l_s/g^2 g_s$ where $g_s$ and $l_s$ are IIA string coupling and length respectively. It was argued in [15, 16] that integrable structure behind the solution of the theory implies that there are $N_c$ D0 branes living on D4 one per each whose equilibrium positions are

$$x_{6,k} = k l_s / N_c g^2. \tag{9}$$

Being lifted to the M theory D0 branes represent the linear bundle on the spectral curve of the integrable system, which has been identified with the Riemann surface $\Sigma$ which

M5 brane is wrapped around. Linear bundle yields half of the phase space of the integrable system and the fluctuations of interacting D0 branes around these points define Toda dynamics which linearizes on the Jacobian of the spectral curve. These branes can be treated as the point-like abelian instantons in $4d$ theory and their possible role was discussed in [16 – 18].

Let us discuss now fermions on the M5 worldvolume. Consider the zero mode of $6d$ Dirac operator $D$ which we decompose as

$$D = D^4 + d^{\Sigma}, \tag{10}$$

where $d^{\Sigma}$ denotes the Dirac operator on the spectral curve. We decompose fermion wave function after IIA projection as

$$\Psi(x, x_6) = \sum_{\lambda} \Psi_{\lambda}(x)\Phi(n, \lambda), \tag{11}$$

where $x = (x_0, x_1, x_2, x_3)$ and $n$ denotes the sites where $D0's$ are localized. We conjecture that $\Phi(n, \lambda)$ is just the Baker function for the Lax equation in Toda system if the eigenvalue of the four dimensional Dirac operator $D^4$ equals to $\lambda$. Baker fermions in the Toda system have $\Sigma$ as the Fermi surface.

In IIA projection one has the fermion zero modes localized on $D0$'s so it is natural to consider fermion wave function $\Phi(n, \lambda)$ with discrete $n$. Fermion zero modes on the nearest $D4$'s overlap and the Dirac operator along $x_6$ acquires the discrete form

$$c_n \Phi_{n-1} + p_n \Phi_n + c_{n+1} \Phi_{n+1} = \lambda \Phi_n; \quad c_n = \exp(x_{n+1} - x_n), \tag{12}$$

where $x_i$ is the position of $i$-th $D0$ brane along $x_6$ direction, and coincides with the familiar expression for the Lax equation in the Toda system in $2 \times 2$ representation.

Let us emphasize that the Hitchin like dynamical system involving $D0$ branes cannot be seen at the classical level and has to be regarded as the quantum or quasiclassical effect. Phase space of the relevant Hitchin system is the hyperkahler manifold and can be interpreted as a hidden Higgs branch of the moduli space in $N = 2$ theories. The very role of the integrability is to restrict $N_c$ KK modes in $M$ theory picture to the $M5$ brane worldvolume. Comparing expressions for the condensate in $N = 1$ theory we expect the identification

$$w(v)dv \propto \frac{\rho(\lambda)}{\lambda} d\lambda. \tag{13}$$

It is worth remarking on the corresponding deformation of the integrable system. At the first step we should fix all integrals of motion in the Toda system and then perturb it by the mass term. It is not clear at the moment if some integrable dynamics survives but the potential degrees of freedom namely $D0$'s still play important role in generation of the superpotential [18]. Therefore one has the proper candidates for the phase space-rational spectral curve which $M5$ brane is wrapped around and the linear bundle on it.

**4.** Some insights come from the analogy with the discrete Peierls model [20], which is relevant for the description of $1d$ superconductivity. The model is defined as follows; there is a periodic $1d$ crystal with $N_c$ sites and fermions interacting with phonon degrees of freedom. It is assumed that fermions are strongly coupled to the lattice and can jump only between the nearest sites. This system is described by the periodic Toda lattice [21], where the Toda Lax operator serves as the Hamiltonian for fermions and Toda degrees of freedom amounts from the phonons.

Spectral curve for the Toda system coincides with the dispersion law of the fermions and Lax equation – with the Shrödinger one. Initial condition for the Toda evolution is chosen dynamically to minimize the total energy of fermions and phonons and it appears that the generic curve degenerates to the "$N = 2$ curve" with all monopole cycles vanish. This resembles the first step towards the $N = 1$ theory. Fermions develop the mass gap and the analogue of the chiral invariance known in the model is broken dynamically due to the nontrivial band around zero energy. This yields a dynamical scenario which resembles one in the $N = 1$ theory.

To conclude, we have considered the possible meaning of the Banks – Casher relation in the MQCD framework. We obtain qualitative picture explaining the nonvanishing spectral density of the Dirac operator at the origin and discuss the geometrical meaning of the eigenvalues themselves. We conjecture that six dimensions involved in the brane configuration for $N = 1$ theory allow interpretation as the Dirac operator eigenvalue and eigenvalues of the two global symmetry generators. Certainly more quantitative analysis is required to confirm the picture suggested.

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1. S.Elitzur, A.Giveon, D.Kutasov, and E.Rabinovichi, Nucl. Phys. **B505**, 202 (1997), (hep-th/9704104).
2. K.Hori, H.Ooguri, and Y.Oz, Adv. Theor. Math. Phys. **1**, 1 (1998), (hep-th/9706082).
3. E.Witten, Nucl. Phys. **B507**, 658 (1997), (hep-th/9706109).
4. S.Nam, K.Oh, and S.Sin, (hep-th/9707247).
5. E.Witten, Nucl. Phys. **B474**, 343 (1996), (hep-th/9604030).
6. C.Gomez, (hep-th/9711074).
7. J.Brodie and A.Hanany, Nucl. Phys. **B506**, (1997), (hep-th/9704043).
8. M.Shifman and G.Dvali, Phys. Lett. **B396**, 64 (1997), (hep-th/9612128). A.Kovner, M.Shifman, and A.Smilga, (hep-th/9706089).
9. T.Banks and A. Casher, Nucl. Phys. **B169**, 103 (1980).
10. H.Leutwyler and A.Smilga, Phys. Rev. **D46**, 5607 (1992).
11. J.Verbarschoot, (hep-th/9710114).
12. E.Witten, (hep-th/9703166).
13. A.Klemm, W.Lerche, P.Mair et al., Nucl. Phys. **B477**, 746 (1996), (hep-th/9604034).
14. A.Gorsky, I.Krichever, A.Marshakov et al., Phys. Lett. **B355**, 466 (1995), (hep-th/ 9505135).
15. A.Gorsky, Phys. Lett. **B410**, 22 (1997), (hep-th/9612238).
16. A.Gorsky, S.Gukov, and A.Mironov, Nucl. Phys. **B517**, 409 (1998), (hep-th/9707120).
17. J.Barbon and A.Pasquinucci, hep-th/9712135; (hep-th/9708041).
18. J.Brodie, (hep-th/9803140).
19. A.Karch, D.Lust, and D.Smith, (hep-th/9803232).
20. A.Gorsky, Mod. Phys. Lett. **A12**, 719 (1997).
21. S.Brazovsky, I.Dzyaloshinsky, and I.Krichever, ZhETF **83**, 389 (1982).