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CLASSIFICATION OF SINGULAR POINTS IN POLARIZATION FIELD OF COSMIC MICROWAVE BACKGROUND AND EIGENVECTORS OF STOKES MATRIX

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Analysis of the singularities of the polarization field of CMB, where polarization is equal to zero, is presented. It is found that the classification of the singular points differs from the usual three types known in the ordinary differential equations. The new statistical properties of polarization field are discussed, and new methods to detect the presence of primordial tensor perturbations are indicated.

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In the coming years the measurements of the angular anisotropies of the intensity of cosmic microwave background (CMB) by the cosmic missions MAP and PLANCK will possibly present one of the most promising methods of studying early universe as well as of precise measuring of basic cosmological parameters(see e.g. [1] and references therein). In addition to the anisotropies of the intensity, it is possible, though more difficult, to measure polarization of the radiation. The polarization is a secondary effect induced by the scattering of anisotropic radiation field on electrons in cosmic plasma. The corresponding measurements of the CMB polarization are planning to be performed in the coming space missions.

Polarization field is described by a traceless 2 × 2-matrix which can be decomposed into a sum of three Pauli matrices σ_i with the coefficients known as Stokes parameters

$$\mathbf{a} = \xi_i \sigma_i. \tag{1}$$

As is well known circular polarization does not arise in Thomson scattering (because of parity conservation), so that $\xi_2 = 0$ and the matrix $\mathbf{a}$ is symmetric. Usually it is parameterized in the form:

$$\mathbf{a} = \begin{pmatrix} Q & U \\ U & -Q \end{pmatrix}. \tag{2}$$

The sources of polarization are anisotropies of radiation field induced by different types of perturbations, namely scalar, tensor and vector ones. Vector perturbations decay in the early universe but may arise at small scales at later stages and influence the CMB polarization in the case of reionization.

Geometrical properties of the polarization field allow to obtain an important cosmological information [2 – 9]. An importance of the study of singular points of polarization field, where $Q = U = 0$, was emphasized in ref. [10]. In this paper geometric classification and statistics of the singular point was proposed. This has been done in terms of the fields Q and U which directly enter polarization matrix. The latter are 2-dimensional (2D) tensor fields and have the appropriate transformation properties under rotation of coordinate system.

In this paper we will investigate the classification of singular points of eigenvectors of the polarization matrix $\mathbf{a}$. Though the positions and statistics of the singular points remains the same, their types could be quite different. The eigenvalues are easily found:

$$\lambda_{1,2} = \pm\sqrt{Q^2 + U^2} \tag{3}$$

and the eigenvector corresponding to the positive λ is

$$\mathbf{n}^{(1)} \equiv (n_x, n_y) \sim \left(U,\ \sqrt{Q^2 + U^2} - Q\right). \tag{4}$$

This vector determines direction of maximum polarization and up to a normalization factor coincides with the direction of the vector $\mathbf{P}$ considered in refs. [4, 6, 7] or orthogonal to it, depending upon the sign of the coefficient functions.

The behavior of the vector field $\mathbf{n}^{(1)}$ in the vicinity of the singular points of polarization, $Q = U = 0$, is determined by the equation

$$\frac{dy}{dx} = \frac{n_y}{n_x} = \frac{\sqrt{Q^2 + U^2} - Q}{U}. \tag{5}$$

Analysis of singular points of differential equations when both numerator and denominator can be expanded into Taylor series is well known and can be found e.g.in ref. [11]. In the usual case the following singular points can exist: focus, saddle, and knot. In our case the situation is more complicated due to the presence of the square root in the numerator which is generically non-analytic in the points where $Q = U = 0$. At this stage a question may arise why it is assumed that Q and U are analytic functions expandable into Taylor series (at least up to first order terms) around the points where $Q = U = 0$, while the component of the eigenvectors are not. The reason for that is the following. The matrix elements of the polarization matrix Q and U are directly related to the anisotropy of the cosmic microwave radiation through the amplitude of photon-electron scattering. We do not expect any grounds for these quantities to have a square root singularity where their first derivative tends to infinity. On the other hand the eigenvectors of the matrix **a** just mathematically contain the square root $\sqrt{Q^2+U^2}$ and so it is singular at $Q=U=0$. The analysis of singularities of the vector field $\mathbf{n}^{(1)}$ can be done as follows. We assume that Q and U are expanded near singular points as

$$Q = a_1 x + a_2 y, \qquad U = b_1 x + b_2 y. \tag{6}$$

In the case when the matrix

$$\mathbf{M} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \tag{7}$$

is not degenerate, det $\mathbf{M} \neq 0$, it is convenient to introduce the new coordinates:

$$\begin{gathered} \xi = a_1 x + a_2 y, \;\; \eta = b_1 x + b_2 y, \\ x = A_1 \xi + A_2 \eta, \;\; y = B_1 \xi + B_2 \eta. \end{gathered} \tag{8}$$

Evidently the types of the singular points do not change under this coordinate transformation. It is simpler to make the further analysis in polar coordinates:

$$\xi = \rho \cos\phi, \;\; \eta = \rho \sin\phi. \tag{9}$$

Eq. (5) in this new polar coordinates can be rewritten as

$$\frac{d(\ln\rho)}{dt} = \frac{2}{t^2+1}\frac{N}{D}, \tag{10}$$

where $t = \tan(\phi/2)$ and

$$N = -A_2 t^3 + t^2(B_2 - 2A_1) + t(2B_1 + A_2) - B_2 \tag{11}$$

and

$$D = A_1 t^3 - t^2(B_1 + 2A_2) + t(2B_2 - A_1) + B_1. \tag{12}$$

Barring the degenerate case of $A_1 = 0$ we may take $A_1 = 1$ without loss of generality.

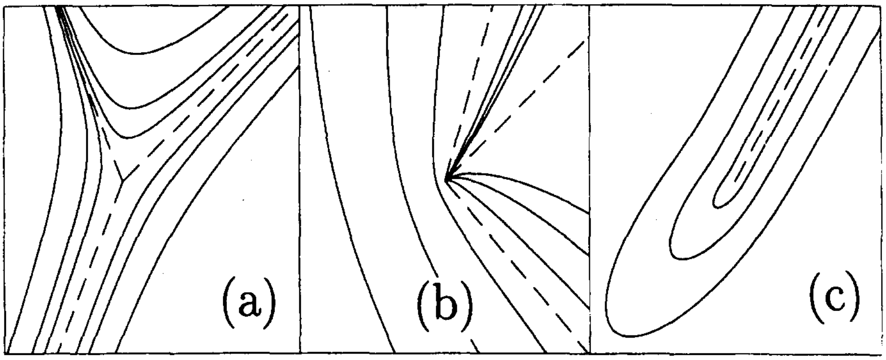
The behavior of singular points depends upon the roots of denominator D. Let us first consider the case when it has three real root, $t_{1,2,3}$. The solution of eq. (10) in this case can be written as:

$$\frac{r}{r_0} = (t^2+1)\prod_j \mid t - t_j \mid^{2\nu_j}, \tag{13}$$

where r_0 is an arbitrary constant and the powers ν_j are

$$\nu_1 = -\frac{(1+t_1^2)(1+t_2 t_3)}{2(t_1-t_2)(t_1-t_3)} \tag{14}$$

and so on by cyclic permutation of indices. Since it can be shown that $\sum \nu_j = -1$, the points where $t^2 \to \infty$ are not generally singular. It can be easily checked that either all $\nu_j < 0$ or any two of them are negative and one is positive. In the first case the singular



Integral curves for three different types of singular points: a) saddle, b) beak, & c) comet. Long dashed lines show peculiar solutions (separatrises)

point resembles the usual saddle with the only difference that there are three and not four, as in the usual case, linear asymptotes/separatrices (see Figure a). We will call it also "saddle". If one of ν_j is positive (say $\nu_1 > 0$), and thus r becomes zero at $t = t_1$, the behavior of the direction field near this point is quite different from the usual ones. The field line cannot be continued along $\phi = \phi_1$ into $\phi = \phi_1 + \pi$ as can be done in the usual case. We will call this type of singularity a "beak" (see Figure b).

In the case of one real root of denominator D the solution has the same form as (13) but now e.g. the powers ν_2 and ν_3 are complex conjugate. The solution can be written as

$$r/r_0 = (t^2+1) \mid t - t_2 \mid^{4\mathrm{Re}\,\nu_2} \exp\left(4\beta \,\mathrm{Im}\,\nu_2\right)(t-t_1)^{2\nu_1}\,, \tag{15}$$

where $\beta = \tan^{-1}[\mathrm{Im}t_2/(t - \mathrm{Re}t_2)]$. The real root ν_1 is negative, as is seen from eq. (14) and thus r does not vanish in vicinity of such singular point. The polarization direction field for this case is presented in Figure c. This type of singularity can be called a "comet".

One can estimate the density of different singular points in the following way (see e.g. [4, 10]). All singular points correspond to the case when both $Q = 0$ and $U = 0$. The number density of these points is proportional to

$$dQdU = \det\begin{pmatrix} Q_x & Q_y \\ U_x & U_y \end{pmatrix} dxdy \tag{16}$$

and thus the density is given by the average value of the determinate, $d = Q_xU_y - Q_yU_x$. It can be easily checked that the saddle-type singularity takes place if $d > 0$ that is the same condition as for normal saddles in the field determined by the equation $dy/dx = Q/U$ see [10]. It can be shown that saddles make 50% of all singular points $\langle n_s\rangle = 0.5\langle n\rangle$, where n is the number density of all singular points. Calculations of the number of beaks and comets are more complicated and can be found numerically. According our estimates the surface densities for beaks and comets are correspondingly $\langle n_b\rangle \approx 0.052\langle n\rangle$ and $\langle n_c\rangle \approx 0.448\langle n\rangle$. We note that the probability of appearance of saddles, beaks and comets for random choice of Q_x, Q_y, U_x,U_y, is correspondingly $W_s = 0.500$, $W_b \approx 0.116$, $W_c \approx 0.384$.

As we have already mentioned the polarization of cosmic microwave radiation arises due to anisotropy of the radiation field. It is a linear functional of the field and in the case of scalar perturbations the only way to construct two dimensional tensor quantity is to use second derivatives of a scalar function Ψ, as is argued e.g. in ref. [9, 12]. The matrix elements of traceless symmetric matrix (2) are constructed uniquely as

$$a_{ij} = 2\partial_i\partial_j\Psi - \delta_{ij}\partial^2\Psi. \tag{17}$$

For the functions Q and U it gives:

$$Q = (\partial_x^2 - \partial_y^2)\Psi, \qquad U = 2\partial_x\partial_y\Psi. \tag{18}$$

In principle one may use also the invariant two-dimensional antisymmetric (pseudo)-tensor ϵ_{ij} but parity conservation prevents from its appearance in polarization matrix in the case of scalar perturbations. For tensor perturbations there could be specific "external" directions in the space and parity considerations do not prevent from using ϵ_{ij} in the matrix (2) (see below). For this specific form (18) of polarization matrix there exist a particular (pseudo)scalar quantity which vanishes in the absence of gravitational perturbations [6, 7]:

$$B = \epsilon_{ij}\partial_k\partial_i M_{kj}. \tag{19}$$

Written explicitly it reads

$$B = (\partial_x^2 - \partial_y^2)U - 2\partial_x\partial_y Q. \tag{20}$$

It evidently vanishes for Q and U given by expressions (18).

In the case when tensor perturbations are present, there is more freedom in polarization matrix **a** and the terms proportional to ϵ_{ij} are permitted and the symmetric matrix **a** may be expressed through second derivatives of two independent scalar functions:

$$M_{ij} = 2\partial_i\partial_j\Psi - \delta_{ij}\partial^2\Psi + \epsilon_{ik}\partial_k\partial_j\Phi\ \epsilon_{jk}\partial_k\partial_i\Phi. \tag{21}$$

This is a general decomposion of symmetric traceless tensor in 2 dimensions (see e.g. [6]).

With inclusion of tensor perturbations the functions Q and U take the form:

$$\begin{aligned} Q &= (\partial_x^2 - \partial_y^2)\Psi + 2\partial_x\partial_y\Phi, \\ U &= 2\partial_x\partial_y\Psi - (\partial_x^2 - \partial_y^2)\Phi. \end{aligned} \tag{22}$$

The scalar B is expressed through the fourth derivatives of Φ as following:

$$B = \partial^4\Phi. \tag{23}$$

The difference between scalar and tensor perturbations appears only in the fourth derivatives of the generating scalar functions Φ and Ψ, while the structure of their singularities is determined by the third derivatives. Thus the types of the singularities are the same for both types of perturbations. There are statements in the literature that in the case of scalar perturbations vector **n** cannot have curl, on the other hand tensor perturbations do produce a curl, see [13], section 4.

However the polarization tensor is not a vector but a second rank tensor and direct analogy with a vector field is not applicable. In the general case of singular points considered above the curl is not equal to zero for any type of perturbations. An explicit example of scalar generating function Ψ being a function only of $r = \sqrt{x^2 + y^2}$ near the singularity point shows that the latter may be either center or knot. These points are absent in our list of three presented above due to a specific degeneracy of this example.

Thus to summarize, the singular points of the vector field $\mathbf{n}^{(1)}$, which is the eigenvector of polarization matrix corresponding to the direction of maximum polarization can be generically of the following three types (see above): saddle and beak (with three separatrices), and comet (one separatrix). In degenerate case, when some of the coefficients or their combinations (like e.g. determinants) may be zero, then there could be some other types singularities which we have not considered here.

The functions Q and U are usually assumed to be independent Gaussian variables with equal dispersion [4]:

$$\langle Q^2 \rangle = \langle U^2 \rangle = \sigma_0^2. \tag{24}$$

Their first derivatives are also independent and non-correlated with the functions with dispersion

$$\langle Q_i Q_j \rangle = \langle U_i U_j \rangle = \delta_{ij}\sigma_1^2/2, \tag{25}$$

where $Q_i = \partial Q/\partial x^i$, etc. All other correlators are zero. However the fact that in the case of scalar perturbations both functions Q and U as well as their derivatives are determined by a single generating scalar function Ψ imposes some conditions on the correlators of the second derivatives. In particular the dispersions of the second derivatives Q_{ij} and U_{ij} are not equal and these fields are correlated. The list of nontrivial correlators is the following:

$$\begin{gathered}
\langle Q_{xx}^2 \rangle = \langle Q_{yy}^2 \rangle = \frac{7}{16}\sigma_2^2, \;\; \langle U_{xx}^2 \rangle = \langle U_{yy}^2 \rangle = \frac{5}{16}\sigma_2^2, \\
\langle Q_{xx} Q_{yy} \rangle = \langle Q_{xy} Q_{xy} \rangle = \frac{1}{16}\sigma_2^2, \;\; \langle U_{xx} U_{yy} \rangle = \langle U_{xy} U_{xy} \rangle = \frac{3}{16}\sigma_2^2, \\
\langle Q_{xx} U_{xy} \rangle = \langle Q_{xy} U_{xx} \rangle = \frac{1}{16}\sigma_2^2, \qquad \langle Q_{yy} U_{xy} \rangle = \langle Q_{xy} U_{yy} \rangle = -\frac{1}{16}\sigma_2^2, \\
\langle Q_{xx} Q \rangle = \langle Q_{yy} Q \rangle = \langle U_{xx} U \rangle = \langle U_{yy} U \rangle = -\sigma_1^2/2,
\end{gathered} \tag{26}$$

where $\sigma_2^2 \equiv \langle (Q_{xx} + Q_{yy})^2 \rangle = \langle (U_{xx} + U_{yy})^2 \rangle$. Note that the asymmetric correlators in the third line are non-vanishing.

These properties permit in principle to discriminate and detect a contribution of tensor (or vector) perturbations into polarization of CMB by measuring the dispersion of second derivatives of the Stokes parameters. In particular for pure scalar perturbations one should expect

$$\langle Q_{xx}^2 \rangle / \langle U_{xx}^2 \rangle = \langle Q_{yy}^2 \rangle / \langle U_{yy}^2 \rangle = 7/5, \quad \langle (U_{xx} - U_{yy})^2 \rangle = 4 \langle Q_{xy}^2 \rangle. \tag{27}$$

A deviation from this number would indicate a contribution from other, different from scalar, types of perturbations. The last equality in (27) corresponds to $B = 0$, see Eq.(20). The property $B = 0$ as a test for the absence of scalar perturbations was indicated in previous works, see for example [7].

An interesting quantity which allows one to relate global characteristics of random field to local properties is the Euler characteristic, χ_E, see [14]. As it was noted in ref. [15] this value is closely linked to the critical value of the amplitude of polarization, $P = \sqrt{Q^2 + U^2}$, for which the regions of high polarization percolate. According to this [14] the percolation begins at the amplitude of polarization corresponding to $\chi_E = 0$.

This critical amplitude was estimated in ref. [10] where it was found that percolation takes place for $p = 1$, where $p = P/\sigma_0$ is the dimensionless amplitude of polarization with unit dispersion. We estimated this quantity in somewhat different way than it was done in ref. [10]. In the 2D case the required value is defined by an equation

$$\chi_E \propto \langle p_{xx} + p_{yy} \rangle p \exp(-p^2/2) \propto (p^2 - 1) \exp(-p^2/2) \tag{28}$$

that is identical to result obtained by ref. [10]. Though the statistical distribution we used is different from the one of ref. [10], we got the same result: percolation occurs at $p = 1$. Let us remind that for the 2D Gaussian fields $\chi_E \propto p \exp(-p^2/2)$ and percolation occurs at $p = 0$. As follows from (17) – (22) the same results are valid for all three types of perturbations of polarization field. This topics will be discussed in more detail in [16].

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