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HAMILTONIAN AVERAGING AND INTEGRABILITY IN NONLINEAR SYSTEMS WITH PERIODICALLY VARYING DISPERSION

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Applying Hamiltonian averaging and quasi-identical-like transformation we demonstrate that the averaged dynamics of high-frequency nonlinear wave in systems with periodically varying dispersion can be described in a particular limit by the integrable nonlinear Schrödinger equation.

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Propagation of high-frequency large amplitude wave in media with varying dispersion is a rather general nonlinear problem with a wide area of physical applications like, for instance, optical pulse transmission in dispersion-managed fiber lines [1], a stretched pulse generation in mode-locking fiber laser systems [2], propagation of high intensities beams in second order nonlinear media with periodic poling, evolution of soliton in a periodically modulated nonlinear waveguide and other applications. Optical pulse transmission in fiber is one of the most bright demonstration of practical application of the fundamental soliton theory. The traditional path-averaged optical soliton preserves its cosh-type shape during propagation by compensating in average the fiber dispersion through nonlinearity. This is possible because the pulse power oscillations (due to periodic amplification of the pulse to compensate for the fiber loss) are very fast. Rapid oscillations of the power can be averaged out and, as a result, the slow pulse dynamics in the traditional soliton-based transmission lines, is governed by the integrable [3] nonlinear Schrödinger equation (NLSE). Interability of the NLSE makes possible to apply well-developed powerful mathematical method of the inverse scattering transform [3] to a variety of practical problems (see e.g. [4 – 7] and references therein). Experimental (and even first commercial [8]) realizations of the multichannel soliton transmission have stimulated further research in soliton theory. In this paper we apply Hamiltonian averaging and quasi-identical transformation to demonstrate that the averaged dynamics of high-frequency nonlinear wave in systems with periodically varying dispersion can be described in some particular limits by the integrable NLSE. As a specific physical and practical application, in the present paper we focus on dispersion-managed soliton transmission. The dispersion-managed (DM) periodic, breathing, soliton-like pulse that stably propagates in fiber system with large variations of the dispersion differs substantially from the fundamental (NLSE) soliton [9 – 29]. There are two scales in the DM pulse propagation: the first (fast dynamics) corresponds to rapid oscillations of the pulse width and power due periodic variations of the dispersion and periodic amplification; and the second (slow dynamics) occurs due to the combined effects of nonlinearity, residual dispersion and averaged effects. Traditional soliton solution of the NLSE with uniform dispersion and without loss realizes continuous balance between nonlinearity and dispersion. Losses and variations of dispersion make impossible in general case to support such balance continuously. Nevertheless, a balance between nonlinear effects and dispersion can be achieved *in average* over the compensation period. As a result, slow dynamics of the DM soliton can be described by the propagation equation averaged over fast oscillations [1, 12]. The DM pulse dynamics typically depends on many system parameters and is rather complicated. Different theoretical approaches have already been developed to describe properties of DM soliton: variational approach [12 – 20] or more advanced root-mean-square momentum method [1, 21], multiscale analysis [22 – 24] methods using averaging [12, 26, 1, 29, 25], including averaging in the spectral domain [12, 13], expansion of DM soliton in the basis of the chirped Gauss – Hermite functions [27, 28, 1].

Because of the practical importance of this problem, it is of evident interest to develop different theoretical methods to describe the main properties of the basic model in different limits. A variety of complimentary mathematical methods can be advantageously used to find an optimal and economical description of any specific practical application. In this paper, using Hamiltonian averaging and quasi-identical-like transform [30] we demonstrate that in some specific limits (including, in particular, a weak dispersion map [26]) the DM soliton is described by the integrable NLSE.

Evolution (in z) of a high-frequency wave in medium with periodically varying dispersion and nonlinearity is governed by the NLSE with periodic coefficients $d(z)$ and $c(z)$ (we assume here that both have the same period) that can be written in the Hamiltonian form:

$$i\frac{\partial A}{\partial z} = \{A, H\} = \frac{\delta H}{\delta A^*} = -d(z)A_{tt} - \epsilon\, c(z)|A|^2 A, \tag{1}$$

with the Hamiltonian

$$H = \int \left\{ d(z)\,|A_t|^2 - \varepsilon \frac{c(z)}{2}\,|A|^4 \right\} dt \tag{2}$$

and the Poisson brackets defined as

$$\{F, G\} = \int \left(\frac{\delta F}{\delta A(t,z)} \frac{\delta G}{\delta A^*(t,z)} - \frac{\delta F}{\delta A^*(t,z)} \frac{\delta G}{\delta A(t,z)} \right) dt. \tag{3}$$

In Eq. (1) the distance z iz normalized by the compensation period L, $d(z) = \tilde{d} + \langle d \rangle$ ($\langle \tilde{d} \rangle = 0$) describes varying dispersion and $c(z)$ corresponds to power oscillations (due to loss and amplification). For notations we refer to our previous papers [1, 27, 13]. Small parameter $\epsilon = L/Z_{NL}$ where L is a compensation period and Z_{NL} (see e.g. [13, 1])

is a characteristic nonlinear scale. True DM soliton presents a solution of Eq. (1) of the form $A(z,t) = \exp(ikz)M(z,t)$ with a periodic function $M(z+L,t) = M(z,t)$. The DM soliton can be viewed as a kind of nonlinear Bloch wave using the terms of the solid state physics. The goal of the theoretical analysis is to present a systematic way to describe family of solutions M with different k. The basic idea suggested in [1, 12] is to use a small parameter ϵ to derive path-averaged model that gives systematic, leading order description of DM soliton. Averaging cannot be performed directly in Eq.(1) because of the large variations of $\tilde{d} \gg \langle d \rangle$. However, path-averaged propagation equation can be obtained in the frequency domain [12, 13]. The approach developed in [12] can be considered as a decomposition of DM pulse dynamics in the fast evolution of the phase and a slow evolution of the amplitude. The shape of the DM soliton then is given by nonlocal nonlinear equation, steady state solutions of which give leading order approximation of DM solitons. In this paper we show that in some limits an averaged equation can be transformed to the *integrable* NLSE. First, let us following [12, 13] make Fourier transform

$$A(t,z) = \int A_\omega \exp[-i\omega t]\, d\omega (\text{here } A_\omega = A(\omega, z)),$$

and re-write basic equation in the frequency domain.

The equation (1) then takes the form

$$i\frac{\partial A_\omega}{\partial z} - d(z)\,\omega^2\, A_\omega + \epsilon \int F_{\omega 123}(z)\,\delta(\omega+\omega_1-\omega_2-\omega_3)\, A_1^*\, A_2\, A_3 d\omega_1 d\omega_2 d\omega_3 = 0, \quad (4)$$

where $F_{\omega 123} = c(z)$. To eliminate the periodic dependence of the linear part we apply following [12, 13] the so-called Floquet – Lyapunov transformation [30]

$$A_\omega = \phi_\omega \exp\{-i\,\omega^2 R_0(z) - i\,\theta(\omega)\}, \quad dR_0/dz = d(z) - \langle d \rangle. \quad (5)$$

We have included here the phase factor $\theta(\omega)$, that does not change the z-dependence of the coefficients. The aim of this transformation is to eliminate the large coefficient $\tilde{d}$ from (1). In the new variables the equation has the form

$$i\frac{\partial \phi_\omega}{\partial z} - \langle d \rangle\,\omega^2\,\phi_\omega + \epsilon \int G_{\omega 123}(z)\,\delta(\omega+\omega_1-\omega_2-\omega_3)\,\phi_1^*\,\phi_2\,\phi_3\, d\omega_1 d\omega_2 d\omega_3 = 0, \quad (6)$$

here $G_{\omega 123}(z) = c(z)\exp\{i\Delta\Omega R_0(z) + i\Delta\theta\}$ and $\Delta\Omega = \omega^2 + \omega_1^2 - \omega_2^2 - \omega_3^2$, $\Delta\theta = \theta_\omega + \theta_1 - -\theta_2 - \theta_3$. Note that $G_{\omega 123}$ depends only on the specific combination of the frequencies given by the resonanse surface $\Delta\Omega$. Both the Fourier and the Floquet – Lyapunov transform (5) are canonical and the transformed Hamiltonian H is

$$H = \langle d \rangle \int \omega^2\,|\phi_\omega|^2\, d\omega - \varepsilon \int \frac{G_{\omega 123}}{2}\delta(\omega+\omega_1-\omega_2-\omega_3)\phi_\omega^*\phi_1^*\phi_2\phi_3 d\omega d\omega_1 d\omega_2 d\omega_3. \quad (7)$$

Now we apply Hamiltonian averaging. Let us make the following change of the variables

$$\phi_\omega = \varphi_\omega + \epsilon \int V_{\omega 123}\,\delta(\omega+\omega_1-\omega_2-\omega_3)\,\varphi_1^*\,\varphi_2\,\varphi_3\, d\omega_1 d\omega_2 d\omega_3,$$

$$V_{\omega 123}(z) = i\int_0^z [G_{\omega 123}(\tau) - T_{\omega 123}]d\tau,$$

with

$$T_{\omega 123} = \langle G_{\omega 123} \rangle = \int_0^1 G_{\omega 123}(z)dz = \int_0^1 c(z) \exp\{i\Delta\Omega R_0(z) + i\Delta\theta\}dz. \tag{8}$$

Path-averaged equation has the form

$$i\frac{\partial \varphi_\omega}{\partial t} - \langle d \rangle \omega^2 \varphi_\omega + \epsilon \int T_{\omega 123} \, \delta(\omega + \omega_1 - \omega_2 - \omega_3) \, \varphi_1^* \, \varphi_2 \, \varphi_3 \, d\omega_1 d\omega_2 d\omega_3 = 0. \tag{9}$$

Here $\varphi(\omega)$ is assumed to decay sufficiently fast in order to assure convergence of the integral. This equation first has been derived in [12, 13] using simple physical consideration. Since ϕ_ω varies slowly, in the leading approximation, on the scale of one period, we can neglect their evolution and integrate Eq. (6) over the period placing ϕ_ω outside of the integrals in z. The Hamiltonian averaging introduced here presents a regular way to calculate next order corrections to the averaged model. From the Hamiltonian structure of the starting equation it is clear that the matrix element $T_{\omega 123}$ has the following symmetries (compare with [31])

$$T_{\omega 123} = T_{1\omega 23} = T_{\omega 132} = T^*_{23\omega 1}. \tag{10}$$

In the case of the lossless ($c(z) = c_0 =$ const, see for details [1]) model and two-step dispersion map built from a piece of a fiber with the dispersion $d_1 + \langle d \rangle$ and length l_1 followed by fiber with dispersion $d_2 + \langle d \rangle$ and length $l_2 = 1 - l_1$ ($d_1 l_1 + d_2(1 - l_1) = 0$) the matrix element $T_{\omega 123}$ takes especially simple form

$$T_{\omega 123} = c_0 \, \frac{\sin[\mu \Delta\Omega/2]}{\mu \Delta\Omega/2}. \tag{11}$$

The parameter $\mu = d_1 l_1$ introduced here is a characteristic of the map strength. Strong dispersion management corresponds to large $\mu \gg \langle d \rangle$ and the so-called weak map corresponds to $\mu \ll \langle d \rangle$. We demonstrate below that, in particular, in the limit of small μ the averaged equation (9) can be transformed to the NLSE. Note that the equation (9) possesses the remarkable property. The matrix element $T_{\omega 123} = \Phi(\Delta\Omega) \exp\{i\Delta\theta\}$ is a function of $\Delta\Omega$ and

$$\Phi(0) = \int_0^1 c(z)dz = \langle c \rangle, \quad \Phi'(0) = -i\langle c \, R_0 \rangle = -i \int_0^1 c(z) \, R_0(z)dz \tag{12}$$

on the resonant surface

$$\omega + \omega_1 - \omega_2 - \omega_3 = 0, \quad \Delta\Omega = \omega^2 + \omega_1^2 - \omega_2^2 - \omega_3^2 = 0. \tag{13}$$

This observation allows us to make the following quasi-identical-like transformation, which eliminates the variable part of the matrix element $T_{\omega 123}$

$$\varphi_\omega = a_\omega + \frac{\epsilon}{\langle d \rangle} \int \frac{T_{\omega 123} - T_0}{\Delta\Omega} \, a_1^* \, a_2 \, a_3 \, \delta(\omega + \omega_1 - \omega_2 - \omega_3) \, d\omega_1 d\omega_2 d\omega_3, \tag{14}$$

where $T_0 = \Phi(0) \exp\{i\Delta\theta\}$. This transformation has no singularities. If the integral part in this transform is small compared with a_ω, then in the leading order we get for a_ω

$$i\frac{\partial a_\omega}{\partial t} - \langle d \rangle \omega^2 a_\omega + \epsilon \int T_0 \, \delta(\omega + \omega_1 - \omega_2 - \omega_3) \, a_1^* \, a_2 \, a_3 \, d\omega_1 d\omega_2 d\omega_3 = 0. \tag{15}$$

This is nothing more, but the integrable nonlinear Schrodinger equation written in the frequency domain.

Obviously, this transformation is quasi-identical only if the integral in Eq. (14) is small compared with a_ω. This is not so in general case and that is why, typical DM soliton has form different from cosh-like shape usual for NLSE soliton. However, if the kernel function in Eq. (14) is small

$$S(\Delta\Omega) = \frac{T_{\omega 123}(\Delta\Omega) - T_0}{\Delta\Omega} \ll 1, \tag{16}$$

then the averaged model can be reduced to the NLSE. In other terms, this is a condition on the functions $c(z)$ and $d(z)$ that makes possible quasi-identical transformation is possible. For instance, one can check that for two-step map described above in the limit $\mu \to 0$, this transformation is, indeed, quasi-identical and the path-average model is the NLSE. Thus, we can express (in this limit) solutions of the equation (9), and, consequently, of the original equation (1) via solutions of the NLSE in the explicit form:

$$A(t,z) = \int a_\omega \, e^{\{-i\omega t - i\omega^2 R_0 - i\theta\}} d\omega + \epsilon \int W_{\omega 123}\, a_1^* \, a_2 \, a_3 \, \delta(\omega + \omega_1 - \omega_2 - \omega_3)\, d\omega_1 d\omega_2 d\omega_3 d\omega,$$

here

$$W_{\omega 123}(z) = \left(V_{\omega 123} + \frac{T_{\omega 123} - T_0}{\langle d \rangle \, \Delta\Omega} \right) \exp\{-i\omega t - i\omega^2 \, R_0(z) - i\theta(\omega)\} \tag{17}$$

and a_ω is a solution of the NLSE (15).

The averaging transformation can also be presented as

$$\phi_\omega = \varphi_\omega + \varepsilon \frac{\delta K}{\delta \varphi_\omega} = \varphi_\omega - \varepsilon \, \{K, \varphi_\omega\}. \tag{18}$$

Therefore, this transform can be viewed as the leading order term in the expansion of a canonical exponential (Lie) transformation

$$\phi_\omega = \exp\left[\{\varepsilon K, ...\}\right] \varphi_\omega, \tag{19}$$

with the functional

$$K = \int \frac{V_{\omega 123}}{2} \delta(\omega + \omega_1 - \omega_2 - \omega_3) \varphi_\omega^* \varphi_1^* \varphi_2 \varphi_3 d\omega d\omega_1 d\omega_2 d\omega_3.$$

After averaging the Hamiltonian H takes the form

$$\langle H \rangle = \langle d \rangle \int \omega^2 \, |\varphi_\omega|^2 \, d\omega - \varepsilon \int \frac{T_{\omega 123}}{2} \delta(\omega + \omega_1 - \omega_2 - \omega_3) \varphi_\omega^* \varphi_1^* \varphi_2 \varphi_3 d\omega d\omega_1 d\omega_2 d\omega_3. \tag{20}$$

The quasi-identical transform of the Hamiltonian $\langle H \rangle$ ($T_{\omega 123} \to T_0$) is given by the formula Eq. (19) with a corresponding functional K_1

$$K_1 = \int \frac{T_{\omega 123} - T_0}{2 \langle d \rangle \Delta\Omega} \delta(\omega + \omega_1 - \omega_2 - \omega_3) a_\omega^* a_1^* a_2 a_3 d\omega d\omega_1 d\omega_2 d\omega_3.$$

In conclusion, using Hamiltonian averaging and quasi-identical-like transformation, we have shown that in some specific limits nonlinear wave propagation in system with periodically varying dispersion and nonlinearity can be described by the integrable NLSE.

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