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THE CORRECTION-TO-SCALING EXPONENT IN DILUTE SYSTEMS

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The leading correction-to-scaling exponent ω for the three-dimensional dilute Ising model is calculated in the framework of the field theoretic renormalization group approach. Both in the minimal subtraction scheme as well as in the massive field theory (resummed four loop expansion) excellent agreement with recent Monte-Carlo calculations (H.G.Ballesteros et al., Phys. Rev. **B58**, 2740 (1998)) is achieved. The expression of ω as series in a $\sqrt{\varepsilon}$-expansion up to $O(\varepsilon^2)$ does not allow a reliable estimate for $d = 3$.

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From renormalization group (RG) theory one knows that in the asymptotic region the values of the critical exponents are universal and scaling laws between them hold. There the couplings of the model Hamiltonian describing the critical system have reached their fixed point values. In the nonasymptotic region deviations from the fixed point values are present. They die out according to a universal power law governed by the correction-to-scaling exponent ω. E.g. for the zero field susceptibility the approach from above to the critical temperature T_c is characterized by the so-called Wegner expansion [1]

$$\chi \simeq \Gamma_0 \tau^{-\gamma}\left(1+\Gamma_1\tau^{\omega/\nu}+\Gamma_2\tau^{2\omega/\nu}+\ldots\right), \tag{1}$$

where $\tau = (T - T_c)/T_c$ and the Γ_i are the non-universal amplitudes. γ and ν are the asymptotic values of the susceptibility and correlation length critical exponents. The smaller the exponent ω, the larger is the region where corrections to the asymptotic power laws have to be taken into account. Being even further away from the fixed point it is necessary to consider the complete non linear crossover functions. This exponent has been calculated with high accuracy for the $O(n)$ symmetric model (in particular for the $3d$-Ising model, see Table), but is much less known for the corresponding diluted model. As a result of a $3d$-calculation of the field theoretic functions within the minimal subtraction scheme [2] and a thorough analysis of different methods for calculating critical exponents [3], we are able to present accurate values of the correction-to-scaling exponent for weakly diluted quenched $3d$-Ising model.

The implication of quenched dilution on the critical behavior is a long-standing problem attracting theoretical, experimental and numerical efforts. In the $3d$-Ising model quenched disorder changes the asymptotic critical exponents compared to the pure ones [4, 5]. In principle this statement should hold for arbitrary weak dilution. But in order

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to observe this change one should approach the critical point close enough. The width of this region turns out to be dilution dependent.

In particular Monte-Carlo (MC) calculations of the critical exponents in the dilute $3d$-Ising model are more difficult to perform than for the pure model since they need much larger sizes of lattices [6]. Even then the exponents were found to be non-universal and varying continously with dilution, i.e. they were effective ones [7]. It became clear that a correction-to-scaling analysis is unavoidable and indeed universal exponents were found [8]. Without it one still obtains concentration dependent effective exponents [9].

Values of correction-to-scaling exponent ω as obtained from different methods in dilute and pure $3d$-Ising models

Method	Dilute	Pure
scaling field	0.42[10]	0.87[10]
ε expansion	see text	0.814 ± 0.018[23]
massive RG, $d=3$	0.372	0.799 ± 0.011[23]
min. sub. RG, $d=3$	0.390	0.791
MC	0.37 ± 0.06[8]	0.8 ± 0.1[16]; $0.8 - 0.85$[24]; 0.87 ± 0.09[25]

For the accuracy of our values see text and Figure

The value of the correction-to-scaling exponent ω found in MC calculations from an analysis invoking the first correction term in (1) turned out to be [8]

$$\omega = 0.37 \pm 0.06\,. \tag{2}$$

Thus it is almost half as large as its corresponding value in the pure model (see Table) and this smallness of ω in the dilute case explains its importance for an analysis of the asymptotic critical behavior. It is therefore highly desirable to have an independent quantitative theoretical prediction for the value of the correction-to-scaling exponent in the dilute system.

In theoretical calculations the value of ω found by scaling field RG [10] is $\omega = 0.42$. So far field theoretical RG studies mainly concentrated on the asymptotic values of the leading exponents. Correction-to-scaling exponents have been calculated within massive RG in two loop approximation in Ref. [11] ($\omega = 0.450$) and within the minimal subtraction scheme in three loop approximation in Ref. [12] ($\omega = 0.366$). Here, we improve this value in the massive RG scheme up to four loop order with the result

$$\omega = 0.372 \tag{3}$$

in excellent agreement with (2). In the minimal subtraction scheme we obtain $\omega = 0.390$ remaining with in the bandwidth of MC accuracy.

The critical behavior of the quenched weakly dilute Ising model in the Euclidian space of $d = 4 - \varepsilon$ dimensions is governed by a Hamiltonian with two couplings [13]:

$$\mathcal{H}(\phi) = \int \mathrm{d}^d R \Big\{ \frac{1}{2} \sum_{\alpha=1}^{n} \left[|\nabla \phi_\alpha|^2 + m_0^2 \phi_\alpha^2 \right] - \frac{v_0}{4!} \left(\sum_{\alpha=1}^{n} \phi_\alpha^2 \right)^2 + \frac{u_0}{4!} \sum_{\alpha=1}^{n} \phi_\alpha^4 \Big\}, \tag{4}$$

in replica limit $n \to 0$. Here ϕ_α are the components of order parameter; $u_0 > 0, v_0 > 0$ are bare couplings; m_0 is bare mass.

We describe the long-distance properties of the model (4) in the vicinity of the phase transition point using a field-theoretical RG approach. The results presented in this paper are obtained on the basis of two different RG schemes: the normalization conditions of massive renormalized theory at fixed [14] $d = 3$ and the minimal subtraction scheme [15]. The last approach allows both fixed $d = 3$ calculations [16] as well as an ε-expansion.

In the RG method the change of the couplings u and v under renormalization is described by two β-functions

$$\beta_u(u,v) = \mu \left(\frac{\partial u}{\partial \mu}\right)_0, \qquad \beta_v(u,v) = \mu \left(\frac{\partial v}{\partial \mu}\right)_0, \tag{5}$$

where μ corresponds to the mass in the massive field theory approach and to the scale parameter in the minimal subtraction scheme. The subscript in (5) indicates that the derivatives are taken at constant unrenormalized parameters. The β-functions differ for different RG schemes and in consequence the fixed point coordinates u^*, v^*, defined by the simultanious zeros of both β-functions, are scheme dependent. The asymptotic critical exponents as well as the correction-to-scaling exponent do not depend on the RG scheme and take universal values.

The correction-to-scaling exponent ω is defined by the smallest eigenvalue of the matrix of derivatives of the β-functions

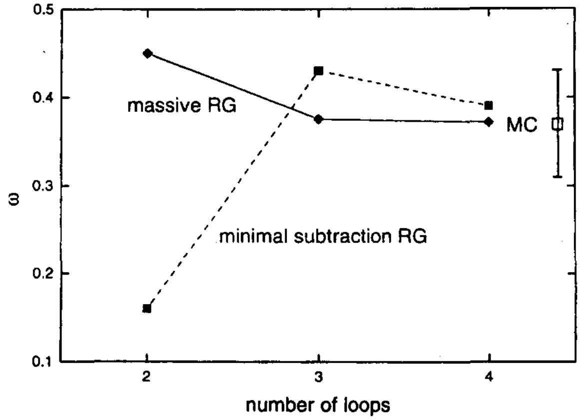
$$\begin{pmatrix} \dfrac{\partial \beta_u}{\partial u} & \dfrac{\partial \beta_u}{\partial v} \\ \dfrac{\partial \beta_v}{\partial u} & \dfrac{\partial \beta_v}{\partial v} \end{pmatrix} \tag{6}$$

taken at the stable fixed point. For the stable fixed point both eigenvalues of this matrix have a positive real part.

Our results for the correction-to-scaling exponent are based on the known high order expansions for the functions β_u and β_v. In the massive scheme they are known in four loop approximation [17]. In the minimal subtraction scheme one can obtain these functions in five loop approximation in the replica limit from those of a cubic model [2]. In the limiting case of the pure model only the coupling u is present. The corresponding β-function results from putting $v = 0$ in $\beta_u(u,v)$ and the correction-to-scaling exponent is simply the derivative $\partial\beta_u(u,0)/\partial u$ taken at the stable fixed point u^*. Note that for the pure model the β-functions in the massive scheme are known in six loop approximation [18] and the five loop results for the RG functions in the minimal subtraction scheme [19] agree with those recovered from Ref. [2].

It is known that the series obtained in the perturbational RG approach are at best asymptotic (for the dilute model see however Ref. [20]). An appropriate resummation procedure has to be applied to the β functions in order to obtain reliable information. The choice of the resummation procedure depends on the information about the high order behavior of the expansion series. This information is not available for the case of the β-functions (5). In this situation we have used in our analysis several resummation procedures. In particular we tried Padé-Borel resummation [18] for resolvent series [21] as well as Chisholm-Borel resummation technique [11, 22]. Simple Padé-tables were analyzed as well. Then, special attention was payed to the choice of the fit parameter (entering Borel-Leroy transform). We observed the standard 'benchmarks', namely fastest convergence of the perturbation theory results, reproductivity of the best accuracy known for exponent values of the pure model. Moreover different forms of the approximants were tried and analyzed on the base of a model function [3].

The steps which we follow in the calculation of the correction-to-scaling exponent ω are the following: First the β-functions (5) are resummed and the system of equations for the fixed points, $\beta_u(u^*, v^*) = 0$, $\beta_v(u^*, v^*) = 0$, is solved. Then the matrix of derivatives (6) is calculated for the resummed β-functions. The stability of the fixed points is checked. The fixed point with both $u^* \neq 0$ and $v^* \neq 0$ is the stable one at $d = 3$ and the smallest eigenvalue gives the desired correction-to-scaling exponent. Note that the eigenvalues might be complex, in this case both have the same positive real part defining ω.



Correction-to-scaling exponent ω of the dilute 3d-Ising model in increasing number of loops. Open square with error bar shows the region of accuracy of the MC data [8]; full squares: our values in the minimal subtraction RG scheme; full diamonds: our values in the massive RG scheme

In Figure we present our results for the exponent ω obtained in successive orders of perturbation theory in number of loops. To perform the resummation the Borel transforms of the truncated lth order perturbation theory expansion for the β-functions were presented in the form of $[(l-1)/1]$ rational approximants of two variables [22]. This form of rational approximants appeared to give the most reliable results. The four loop results for the exponent ω obtained in both RG schems are given in the second column of Table. The behavior of ω in succesive numbers of loops shown in Figure. The uncertainty in ω may be estimated by taking the difference between the four loop and the three loop result. It gives in all cases the typical accuracy of lower then 10%. Although both RG schemes lead to comparable values for ω, the convergence of the values in the massive scheme is much faster. Note that the result for ω combined with the corresponding four loop results for the asymptotic critical exponents [17, 26] confirms the conjectured inequality, $-\nu\omega < \alpha < 0$, for the random models critical exponents involving the specific heat exponent α [27].

As it was noted above five loop results for the minimal subtraction scheme are available [2]. In particular applying the resumation scheme [28] to the pure Ising model case, $v = 0$, we get the following values for ω in increasing number of loops starting from two loop: $\omega = 0.566; 0.852; 0.756; 0.791$. This leads to an improvement in accuracy of the previously calculated $d = 3$ five loop value [16] (see the third column of Table).

The degeneracy of the dilute Ising model β-functions on the one loop level leads to the $\sqrt{\varepsilon}$-expansion [13, 29]. For the critical exponents this expansion is known up to $\mathcal{O}(\varepsilon^2)$ [30]. Starting from the five loop results of Ref. [2] in the replica limit we get the following expansions [31] for the eigenvalues ω_1 and ω_2 of the stability matrix (5) in the fixed point $u^* \neq 0$, $v^* \neq 0$:

$$\omega_1 = 2\varepsilon + 3.704011194\,\varepsilon^{3/2} + 11.30873837\,\varepsilon^2, \quad (7)$$
$$\omega_2 = 0.6729265850\,\varepsilon^{1/2} - 1.925509085\,\varepsilon - 0.5725251806\,\varepsilon^{3/2} - 13.93125952\,\varepsilon^2.$$

From naively adding the successive perturbational contributions one observes that already in three loop approximation ($\sim \varepsilon$) ω_2 becomes negative and therefore **no stable fixed point** exists in strict $\sqrt{\varepsilon}$-expansion. Even the resummation procedures we applied above, do not change this picture [26]. This can be considered as indirect evidence that the $\sqrt{\varepsilon}$-expansion is not Borel summable, as may be expected from Ref. [20]. A physical reason might be the existence of the Griffith singularities caused by the zeros of the partition function of the pure system [32]. The fixed d approach, both within the massive [14] and minimal subtraction [15, 16] schemes, seems to be the only reliable way to study critical behaviour of the model by means of RG technique.

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