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LAGRANGIAN INSTANTON FOR THE KRAICHNAN MODEL

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We consider high-order correlation functions of the passive scalar in the Kraichnan model. Using the instanton formalism we find the scaling exponents ζ_n of the structure functions S_n for $n \gg 1$ under the additional condition $d\zeta_2 \gg 1$ (where d is the dimensionality of space). At $n < n_c$ (where $n_c = d\zeta_2/[2(2-\zeta_2)]$) the exponents are $\zeta_n = (\zeta_2/4)(2n - n^2/n_c)$, while at $n > n_c$ they are n-independent: $\zeta_n = \zeta_2 n_c/4$. We also estimate n-dependent factors in S_n.

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Anomalous scaling is probably the central problem of the theory of turbulence. In 1941 Kolmogorov formulated his famous theory [1], where scaling behavior of different correlation functions of velocity in a turbulent flow was predicted. Experimentally one observes deviations from the scaling exponents, proposed by Kolmogorov. It is recognized that the deviations are related to rare strong fluctuations making the main contribution into the correlation functions [2]. This phenomenon, called intermittency, is the most striking peculiarity of developed turbulence.

One of the classical objects in the theory of turbulence is so-called passive scalar advected by a fluid, the role of the scalar can be played by temperature or by pollutants density. Correlation functions of the scalar in a turbulent flow possess a scaling behavior which in the frame of the theory analogous to that of Kolmogorov was established by Obukhov and Corrsin [3]. Intermittency enforces deviations from the Obukhov exponents that appear to be even stronger than the deviations from the Kolmogorov exponents for the correlation functions of the velocity [4].

A consistent theory of turbulence describing anomalous scaling is not constructed yet because of difficulties associated with the strong coupling inherent to developed turbulence. This is a motivation for attempts to examine the intermittency phenomenon in the framework of different simplified models. The most popular model used for this purpose is the Kraichnan model of the passive scalar advection [5], where the advecting velocity is believed to be short-correlated in time. It allows one to examine the statistics of the scalar in more detail. The scalar in the Kraichnan model exhibits strong intermittency even if such is absent in the advecting velocity. The fact was proved both theoretically [6 – 8] and numerically [9]. In the theoretical works the equation for the n-point correlation function F_n was solved assuming that different parameters, such as ζ_2, $2-\zeta_2$, or d^{-1}, are small. The order of the correlation functions that can be examined in such a way is bound from above. There were several attempts to find the scaling of the correlation functions for larger n. In the work by Kraichnan [10] a closure was assumed enabling to find ζ_n for any n. An alternative scheme was proposed in [11]. An attempt to solve the problem at large n was made in [12].

In the present work we develop a consistent formalism based on the path-integral representation of the dynamical correlation functions of classical fields [13]. The main idea, formulated in [14], is related to the possibility to use the saddle-point approximation in the path integral at large n. The saddle-point conditions are integro-differential equations describing an object that, by analogy with the quantum field theory, we call instanton. The instantonic method was already successfully used in some contexts [15, 16]. The formalism presented in this paper enables to find correlation functions of the passive scalar for arbitrary $n \gg 1$, provided $d\zeta_2 \gg 1$.

Advection of a passive scalar θ by a velocity field v is described by the equation

$$\partial_t \theta + \mathbf{v}\nabla\theta - \kappa\nabla^2\theta = \phi\,, \tag{1}$$

where κ is the diffusion coefficient and ϕ is the source of the scalar. In a turbulent flow v and ϕ are random functions of time and space coordinates. Then passive scalar correlation functions are determined by the statistics of v and ϕ. Usually, one treats simultaneous correlation functions $F_n = \langle \theta(\mathbf{r}_1)\ldots\theta(\mathbf{r}_n)\rangle$, since large-scale velocity fluctuations destroy temporal correlations.

It is convenient to examine the anomalous scaling in terms of the structure functions

$$S_n(\mathbf{r}) = \langle |\theta(\mathbf{r}/2) - \theta(-\mathbf{r}/2)|^n \rangle\ . \tag{2}$$

One expects that in the convective interval of scales the structure functions reveal a scaling behavior $S_n(r) \propto r^{\zeta_n}$. The exponents ζ_n are of great interest since they reflect the intermittency. In the frame of the Obukhov theory [3] $\zeta_n = (n/2)\zeta_2$. Thus the differences $(n/2)\zeta_2 - \zeta_n$ give the anomalous scaling exponents.

In the Kraichnan model v and ϕ are independent random functions δ-correlated in time and described by Gaussian statistics. Therefore, statistical properties of the fields are entirely characterized by their pair correlation functions. For the pumping $\langle \phi(t_1,\mathbf{r}_1)\phi(t_2,\mathbf{r}_2)\rangle = \delta(t_1-t_2)\chi(r_{12})$, where $\chi(x)$ is a smooth function decaying on a scale L. The constant $\chi(0) \equiv P_2$ determines the pumping rate of θ^2. For the velocity field

$$\langle v_\alpha(t_1,\mathbf{r}_1)v_\beta(t_2,\mathbf{r}_2)\rangle = \delta(t_1-t_2)\left\{\mathcal{V}_0\delta_{\alpha\beta} - \mathcal{K}_{\alpha\beta}(\mathbf{r}_1-\mathbf{r}_2)\right\}\ .$$

The quantity $\mathcal{V}_0$ is an $\mathbf{r}$-independent constant which represents the contribution of large-scale velocity fluctuations. Since a homogeneous advection does not influence S_n, one should keep in the correlation function also a small $\mathbf{r}$-dependent correction $\mathcal{K}$ which is assumed to possess some scaling properties:

$$\mathcal{K}_{\alpha\beta}(\mathbf{r}) = \frac{D}{d}r^{-\gamma}\left[r^2\delta_{\alpha\beta} + \frac{2-\gamma}{d-1}(r^2\delta_{\alpha\beta} - r_\alpha r_\beta)\right]\,, \tag{3}$$

where D is a constant characterizing the strength of velocity fluctuations. One assumes that fluctuations of the velocity are strong enough to ensure the large value of the Peclet number, that is $L^{2-\gamma} \gg r_d^{2-\gamma} \sim \kappa/D$. Then there exists a convective interval of scales $r_d \ll r \ll L$. We will be interested in the scaling properties of correlation functions of θ only in this interval, assuming L/r to be the largest parameter in the theory.

Within the Kraichnan model one can derive a closed equation for any correlation function F_n [5, 8, 17]. The equation for F_2 can be solved [5]. In the convective interval $S_2(r) = 2[F_2(0) - F_2(r)] \propto r^\gamma$, which implies $\zeta_2 = \gamma$. However, for $n > 2$ the equations for F_n are too complicated to be solved exactly. In [7, 8] the equations were analyzed in the limits $(2-\gamma) \ll 1$ and $d\gamma \gg 1$. The analysis led to the answer

$$\zeta_n = \frac{n\gamma}{2} - \frac{2-\gamma}{2(d+2)} n(n-2). \tag{4}$$

The first term in the right-hand side of Eq. (4) represents the normal scaling whereas the second one is just the anomalous scaling exponent. The calculations leading to (4) are correct if the anomalous contribution is much smaller than the normal one. To overcome the restriction we proposed a procedure which will be described in detail elsewhere [18]. Below we sketch our scheme.

The diffusivity κ does not enter the expressions for the structure functions in the convective interval [2]. However, it is not possible to put simply $\kappa = 0$, since the diffusion provides an important regularization. Say, two infinitely close fluid particles do not disperse without diffusion, and the average value of θ^2 is infinite, as is seen from Eq. (6). Nevertheless, we put $\kappa = 0$, approximating the structure functions as averages of the powers of the smoothed difference

$$\vartheta = \int dx\, \beta(\mathbf{x})\theta(0,\mathbf{x}), \quad \beta = \delta_\Lambda\left(\mathbf{x} - \frac{r}{2}\right) - \delta_\Lambda\left(\mathbf{x} + \frac{r}{2}\right), \tag{5}$$

where $\delta_\Lambda(\mathbf{x})$ is a function which rapidly tends to zero at $r > \Lambda^{-1} \gg r_d$ and is normalized by the condition $\int dx\, \delta_\Lambda(\mathbf{x}) = 1$. In the absence of diffusion, the regularization is provided by the finite support of δ_Λ.

In the diffusionless case Eq. (1) can be solved in terms of Lagrangian trajectories s:

$$\theta(0,\mathbf{r}) = \int_{-\infty}^{0} dt'\, \phi\left[t', \mathbf{s}(t',\mathbf{r})\right], \quad \partial_t \mathbf{s} = \mathbf{v}(t,\mathbf{s}). \tag{6}$$

The times here are negative due to causality and r is supposed to be the terminating point of the trajectory $\mathbf{s}(t,\mathbf{r})$: $\mathbf{s}(t=0,\mathbf{r}) = \mathbf{r}$. Then we can obtain

$$\langle|\vartheta|^n\rangle = \int \frac{dy\, d\vartheta}{2\pi} \left\langle \exp\left(-\mathcal{F} - iy\vartheta + n\ln|\vartheta|\right)\right\rangle_v, \tag{7}$$

$$\mathcal{F} = \frac{y^2}{2} \int dt\, dr_1\, dr_2\, \chi(R_{12})\beta(\mathbf{r}_1)\beta(\mathbf{r}_2), \tag{8}$$

where $R_{12} = |s(t,\mathbf{r}_1) - s(t,\mathbf{r}_2)|$. The brackets in the right-hand side of Eq. (7) denote averaging over the statistics of $\mathbf{v}$, while the statistics of ϕ is already accounted there.

Note that $\mathcal{F}$ defined by Eq. (8) depends only on the absolute values $R_{12}(t)$ of Lagrangian differences. Therefore, instead of averaging over the statistics of v one may find an answer by averaging over statistics of R_{12} which can be established starting from the relation

$$\gamma^{-1}\partial_t R_{12}^\gamma = \zeta_{12} \equiv R_{12}^{\gamma-2} R_{12\alpha}(v_{1\alpha} - v_{2\alpha}), \tag{9}$$

following from Eq. (6). Using the statistical properties of ζ and employing the conventional procedure [13] we find the effective action

$$i\mathcal{I} = i\int_{-\infty}^{0} dt \int dr_1\, dr_2\, m_{12}(\gamma^{-1}\partial_t R_{12}^\gamma + D) -$$

$$-\frac{D}{d}\int_{-\infty}^{0} dt \int dr_1\, dr_2\, dr_3\, dr_4\, Q_{12,34} m_{12} m_{34}\,, \tag{10}$$

describing the statistics of R_{12}. Here $m_{12} \equiv m(t, \mathbf{r}_1, \mathbf{r}_2)$ is the auxiliary field conjugated to R_{12}. The explicit expression for the function Q is rather cumbersome. We will need it only in the main order over $1/d$

$$Q_{12,34} = \frac{1}{4} R_{12}^{\gamma-2} R_{34}^{\gamma-2} \left(R_{23}^{2-\gamma} + R_{14}^{2-\gamma} - R_{13}^{2-\gamma} - R_{24}^{2-\gamma}\right)\left(R_{23}^2 + R_{14}^2 - R_{13}^2 - R_{24}^2\right)\,. \tag{11}$$

Now, we can rewrite (7) as a path integral

$$\langle|\vartheta|^n\rangle = \int \frac{dy\, d\vartheta}{2\pi} \mathcal{D}R\, \mathcal{D}m\, \exp\left(i\mathcal{I} - \mathcal{F} - iy\vartheta + n\ln|\vartheta|\right)\,. \tag{12}$$

The definition of R leads to the triangle inequalities

$$R_{12} + R_{23} > R_{13}\,, \tag{13}$$

to be satisfied for any three points. Actually, the inequalities are constraints that should be imposed on the field R_{12} when integrating in Eq. (12).

We calculate the integral (12) in the saddle-point approximation regarding the number n to be large enough. Here we will present only results of the calculations to be described in [18].

At $n < n_c$, where

$$n_c = d\gamma/(2(2-\gamma))\,, \tag{14}$$

we obtain

$$S_n \sim \left(\frac{n}{\gamma}\frac{P_2 C_1}{D} L^\gamma\right)^{n/2} \left(\frac{r}{L}\right)^{\zeta_n}\,, \tag{15}$$

$$\zeta_n = n\gamma/2 - (2-\gamma)n^2/(2d)\,. \tag{16}$$

The quantity C_1 in expression (15) is a constant of order unity, whose value depends on the shape of χ (that is on the details of the pumping) and is consequently non-universal. Note that the $\mathbf{r}$-independent factor in (15) is determined by the single-point root-mean square value of the passive scalar $\theta_{rms}^2 \sim P_2 L^\gamma/(D\gamma)$. Comparing expression (16) with (4), we see that they coincide under the conditions $n \gg 1$ and $d \gg 1$ that were implied in our derivation. Surprisingly, the n-dependence of ζ_n given by Eq. (4) is correct not only in the limit $n \ll n_c$, but up to $n = n_c$ which is the boundary value for (15), (16). In the case $n > n_c$ the scaling exponents ζ_n appear to be n-independent and equal to the value $\zeta_c = d\gamma^2/(8(2-\gamma))$. The n-dependent numerical factors can be found in two limits: $n - n_c \ll n_c$ and $n \gg n_c$. At $n \gg n_c$ the structure function are

$$S_n \sim \left(\frac{n}{\gamma}\frac{P_2 C_2}{D} L^\gamma\right)^{n/2} \left(\frac{r}{L}\right)^{\zeta_c}\,. \tag{17}$$

The quantity C_2 in Eq. (17) is again a non-universal constant of order unity.

The vicinity of the critical value $n = n_c$ requires a separate consideration. The expression for the structure functions can be written as

$$S_n \sim \left(\frac{(n-n_c)^2}{\gamma n_c}\frac{P_2 C_\pm}{D} L^\gamma\right)^{n_c/2} \left(\frac{r}{L}\right)^{\zeta_n}\,, \tag{18}$$

which implies the condition $|n - n_c| \ll n_c$. The factors $C_\pm$ are non-universal constants of order unity which are different for the cases $n < n_c$ and $n > n_c$. The exponents ζ_n in expression (18) are determined by Eq. (16) at $n < n_c$ and $\zeta_n = \zeta_c$ at $n > n_c$. The main peculiarity that appears in expression (18) is its critical dependence $\propto |n - n_c|^{n_c}$ which is saturated at very small $n_c - n$. The condition that determines the validity of Eq. (18) is $\gamma \ln L/r \gg n_c/|n - n_c|$.

Let us discuss our results. We have found the n-dependence of the structure function exponents ζ_n which grow with increasing n up to $n = n_c$ and then stop to vary. Our results contradict to the schemes proposed in [10, 11]. The value ζ_c is different and smaller than the constant obtained in [12], that can be considered as an estimate from above. Eq. (16), valid at $n < n_c$, exactly corresponds to the log-normal statistics [19]. The log-normal answer can be obtained if to accept that for a large fluctuation, giving the main contribution into S_n, the pumping is inessential and that the fluctuation is smooth on the scale r. Then, we get from Eq. (1) the passive scalar difference satisfies an equation $\partial_t \ln(\Delta\theta) = -\mathbf{v}\mathbf{r}/r^2$, where we substituted $\nabla\theta$ by $\Delta\theta/r$. From here, as a consequence of the central limiting theorem, we immediately get the log-normal statistics for $\Delta\theta$. The saturation at $n > n_c$ can be explained by the presence of quasi-discontinuous structures in the field θ making the main contribution to the high-order correlation functions of θ. Note also a similar non-analytical behavior of ζ_n for Burgers' turbulence [2] which is explained by presence of shocks in the velocity field. Although formally our scheme is applicable only in the limit $d\gamma \gg 1$, this simple physical picture allows one to hope that the main features of our results persist for arbitrary values of the parameters. This hope is supported by [20], where a saturation of ζ_n was observed in numerical simulations of the Kraichnan model at $d = 3$.

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