Pis'ma v ZhETF, vol.63, iss.12, pp.970 - 973

STATISTICS OF S-MATRIX POLES IN FEW-CHANNEL CHAOTIC SCATTERING: CROSSOVER FROM ISOLATED TO OVERLAPPING RESONANCES

Ya. V.Fyodorov[1)], *H.-Jü.Sommers*

Fachbereich Physik, Universität-Gesamthochschule Essen
45117 Essen, Germany

Submitted 30 May 1996

We derive the explicit expression for the distribution of resonance widths in a chaotic quantum system coupled to continua via M equivalent open channels. It describes a crossover from the χ^2 distribution (regime of isolated resonances) to a broad power-like distribution typical for the regime of overlapping resonances. The first moment is found to reproduce exactly the Moldauer–Simonius relation between the mean resonance width and the transmission coefficient.

PACS: 05.90.+m, 71.23.-k

One of the basic concepts in the domain of chaotic quantum scattering [1, 2] is the notion of resonances, representing long-lived intermediate states to which bound states of a "closed" system are converted due to coupling to continua. On a formal level resonances show up as poles of the scattering matrix $S_{ab}(E)$ occuring at complex energies $E_k = \mathcal{E}_k - \frac{i}{2}\Gamma_k$, where $\mathcal{E}_k$ and Γ_k are called position and width of the resonance, correspondingly.

Whereas the issue of energy level statistics in closed chaotic systems was addressed in an enormous amount of papers statistical characteristics of resonances are much less studied and attracted significant attention only recently [4–7]. In the case of weak effective coupling to continua individual resonances do not overlap: $\langle\Gamma\rangle \ll \Delta$, with Δ standing for the mean level spacing of the "closed" system and $\langle\Gamma\rangle$ standing for the mean resonance width. Under these conditions one can use a simple first order perturbation theory to calculate resonance widths in terms of eigenfunctions of the closed system, see e.g [8, 9]. Quite generally, one finds in such a procedure that the scaled widths $y_s = \frac{\Gamma}{\langle\Gamma\rangle}$ are distributed according to the so-called χ^2-distribution:

$$\rho(y_s) = \frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)} y_s^{\nu/2-1} \exp\left(-\frac{\nu}{2} y_s\right) \tag{1}$$

where $\Gamma(z)$ stands for the Gamma function and the parameter $\nu = M$ ($\nu = 2M$) for systems with preserved (broken) time reversal symmetry (TRS), M being the number of open scattering channels. The case $\nu = 1$ is known as Porter–Thomas distribution and was shown to be in agreement with experimental data (see references in [1, 10] and [11].

Experimentally, one quite frequently encounter the case of only $M \sim 1$ open channels and $\langle\Gamma\rangle \sim \Delta$ [8, 10–12]. The main goal of the present publication is to provide the explicit distribution of resonance widths covering the whole range of the parameter $\langle\Gamma\rangle/\Delta$ for the few channel scattering problem.

[1)] On leave from: Petersburg Nuclear Physics Institute, 188350 Gatchina, St.Petersburg District, Russia.

The starting point for our calculation is the following multichannel Breit – Wigner representation for elements of the scattering matrix [13, 14]:

$$S_{ab}(E) = \delta_{ab} - 2i\pi \sum_{ij} W^{*}_{ia}[E - \mathcal{H}_{ef}]^{-1}_{ij} W_{jb}, \qquad (\mathcal{H}_{ef})_{ij} = H_{ij} - i\pi \sum_{a} W_{ia} W^{*}_{ja}. \tag{2}$$

Here the $N \times N$ matrix H_{ij} is to model the Hamiltonian of a closed chaotic system and thus is chosen to be a member of the corresponding Gaussian ensemble of random matrices. The latter principle is a commonly accepted one in the domain of Quantum Chaos [15]. To be specific, we consider in the present publication only systems with broken TRS by taking H_{ij} to be a random Hermitian matrix from the Gaussian Unitary Ensemble. The amplitudes W_{ai}, $a = 1, 2, ..., M$ are matrix elements coupling the internal motion to one out of M open channels. Without much loss of generality these amplitudes can be chosen in a way ensuring that the average S-matrix is diagonal in the channel basis: $\overline{S_{ab}} = \delta_{ab}\overline{S_{aa}}$. Then one finds the following expression [13]:

$$\overline{S_{aa}} = \frac{1 - \gamma_c g(E)}{1 + \gamma_c g(E)}; \quad \gamma_c = \pi \sum_{i} W^{*}_{ia} W_{ia}, \tag{3}$$

where $g(E) = iE/2 + \pi\overline{\rho(E)}$; $\pi\overline{\rho(E)} = (1 - E^2/4)^{1/2}$ and we assumed that all M channels are statistically equivalent for the sake of simplicity. The strength of coupling to continua is convenient to characterize via the transmission coefficients $T_a = 1 - |\overline{S_{aa}}|^2$. It is easy to see that both limits $\gamma_c \to 0$ and $\gamma_c \to \infty$ equally correspond to the weak effective coupling regime $T_a \ll 1$ whereas the strongest coupling (at fixed energy E) corresponds to $\gamma_c = 1$.

In order to get access to the distribution of resonance widths we use the fact that resonances are actually eigenvalues of the non-Hermitian Hamiltonian matrix $\mathcal{H}_{ef}$ defined in eq.(2). Two-dimensional density of these eigenvalues $\rho(X, Y)$ in the complex plane $E = X + iY$ can be found if one knows the "potential" [5]:

$$\Phi(X, Y, \kappa) = \overline{\frac{1}{2\pi N} \ln \mathrm{Det}[(E - \mathcal{H}_{ef})(E - \mathcal{H}_{ef})^{\dagger} + \kappa^2]}$$

in view of the relation: $\rho(X, Y) = \lim_{\kappa \to 0} \partial^2 \Phi(X, Y, \kappa)$, where ∂^2 stands for the two-dimensional Laplacian. Technically, it turns out to be much easier to restore the potential from its derivative:

$$\frac{\partial^2 \Phi}{\partial \kappa^2} = \frac{1}{2\pi N} \frac{d}{d\kappa} \lim_{\kappa_b \to \kappa} \frac{\partial}{\partial \kappa} \ln Z(\kappa_b, \kappa), \qquad Z(\kappa_b, \kappa) = \frac{\mathrm{Det}[(E - \mathcal{H}_{ef})(E - \mathcal{H}_{ef})^{\dagger} + \kappa^2]}{\mathrm{Det}[(E - \mathcal{H}_{ef})(E - \mathcal{H}_{ef})^{\dagger} + \kappa_b^2]}. \tag{4}$$

Then one can find the following representation for the generating function $Z(\kappa_b, \kappa)$ in terms of the Gaussian integral over both commuting and anticommuting (Grassmann) variables:

$$\begin{aligned} (-1)^N Z(\kappa_b, \kappa) &= \int [d\Psi] \exp\{-\mathcal{L}_0(\Psi) - \mathcal{L}_1(\Psi)\}, \\ \mathcal{L}_0(\Psi) &= \kappa_b(\Psi^{\dagger} \hat{\Lambda} \hat{L} \Psi) + iX(\Psi^{\dagger} \hat{L} \Psi) - i\Psi^{\dagger}(H \otimes \hat{L})\Psi, \\ \mathcal{L}_1(\Psi) &= -Y(\Psi^{\dagger} \hat{\sigma}_0 \Psi) - \Psi^{\dagger}(\hat{\Gamma} \otimes \hat{\sigma}_0)\Psi + (\kappa - \kappa_b)(\Psi^{\dagger} \hat{K} \Psi), \end{aligned} \tag{5}$$

where $\Psi^{\dagger} = (\mathrm{S}_1^{\dagger}, \mathrm{S}_2^{\dagger}, \chi_1^{\dagger}, \chi_2^{\dagger})$; $[d\Psi] = \prod_{p=1,2} d\mathrm{S}_p d\mathrm{S}_p^{\dagger} d\chi_1 d\chi_1^{\dagger}$, with S_p and χ_p being N-component vectors of complex commuting and Grassmannian variables, respectively. The 4×4 matrices $\hat{\Lambda}, \hat{L}, \hat{\sigma}_0$ and $\hat{K}$ are (block)diagonal of the following structure:

$\hat{\Lambda} = \text{diag}(1,-1,1,-1)$; $\hat{L} = \text{diag}(1,-1,1,1)$; $\hat{K} = \text{diag}(0,0,1,-1)$; $\hat{\sigma_0} = \text{diag}(i\Sigma_x, \Sigma_x)$ and $\Sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The entries of the $N \times N$ matrix $\hat{\Gamma}$ are given by $\Gamma_{ij} = -\pi \sum_a W_{ia} W^*_{ja}$.

In order to calculate the ensemble average we apply the well-developed technique pioneered by Efetov [16]. After a set of standard manipulations [13, 16] one arrives at the following expression for the density $\rho(y)$ of widths $y = \pi\Gamma/\Delta$ (normalized to the local mean level spacing $\Delta(X)$ of the closed system) of those resonances, whose positions are within the narrow window around the point X of the spectrum:

$$\rho(y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\int_0^\infty dz z\phi(y,z), \tag{6}$$

where

$$\phi(y,z) = i\frac{\partial}{\partial z}\int d\mu(Q)\text{Str}(\hat{K}\hat{Q})\,\text{Sdet}^{-M/2}\left[1 - \frac{i}{2\tau}(\hat{Q}\hat{\sigma} + \hat{\sigma}\hat{Q})\right] \times \\ \times \exp\{-\tfrac{iz}{2}\text{Str}(\hat{Q}\hat{\Lambda}) - \tfrac{iy}{2}\text{Str}(\hat{Q}\hat{\sigma})\} \tag{7}$$

where $\hat{\sigma} = \text{diag}(\Sigma_x, \Sigma_x)$ and the (graded) matrices $\hat{Q}$ satisfying $\hat{Q}^2 = -1$ are taken from the graded coset space $U(1,1/2)/U(1/1) \otimes U(1/,1)$, whose explicit parametrization can be found in [16]. Here we introduce the notation $\tau = \frac{1}{2\pi\bar{\rho}(X)}(\gamma_c + \gamma_c^{-1})$ and used the symbols Str, Sdet for the graded trace and the graded determinant, correspondingly.

Evaluating the integral over matrices $\hat{Q}$ one arrives at the following explicit expression for the distribution of resonance widths for a M-channel chaotic system with broken time-reversal invariance. It constitutes the main result of the present publication:

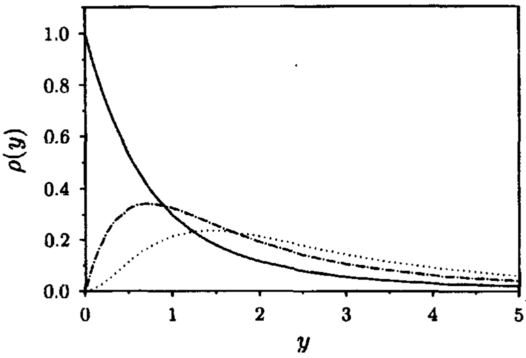
$$\rho(y = \frac{\pi\Gamma}{\Delta}) = \frac{1}{2\Gamma(M)y^2}\int_{y(\tau-1)}^{y(\tau+1)} dt t^M e^{-t} = (-1)^M \frac{y^{M-1}}{\Gamma(M)}\frac{d^M}{dy^M}\left(e^{-y\tau}\frac{\sinh y}{y}\right). \tag{8}$$

Quick inspection of eq.(8) shows that it is indeed reduced to the χ^2 distribution, eq.(1) when the effective coupling to continua is weak: $\tau \gg 1$. Under this condition resonances are typically too narrow to overlap with others: $y \lesssim \tau^{-1} \ll 1$. However, as long as the effective coupling becomes stronger, the parameter τ decreases towards unity. Under these conditions another domain of resonance widths becomes more and more important: $M/(\tau+1) < y < M/(\tau-1)$, where the distribution eq.(8) shows the powerlaw decrease: $\rho(y) \approx \frac{1}{2}My^{-2}$. The most drastic difference from eq.(1) occurs for the maximal effective coupling $\tau = 1$ (i.e $\gamma_c = 1$ and $E = 0$). In this regime the powerlaw tail extends up to infinity, see fig.1, making all positive moments (starting from the first one) to be apparently divergent.

It is interesting to note that the behaviour $\rho(y) \approx \frac{1}{2}My^{-2}$ nicely matches that obtained in [5] for the asymptotic limit $M \propto N \gg 1$. Another point which is worth to be mentioned is that evaluating the first moment of the distribution eq.(8) exactly we arrive at the following expression for the mean resonance widths:

$$\frac{\langle\Gamma\rangle}{\Delta} = -\frac{M}{2\pi}\ln\frac{\tau-1}{\tau+1} \equiv -\frac{M}{2\pi}\ln(1 - T_a). \tag{9}$$

This formula is well known in nuclear physics as Moldauer–Simonius relation [17].



The distribution of scaled resonance widths $\rho(y)$ for $M = 1$ (solid), $M = 2$ (dash-dotted) and $M = 3$ (dotted line) open channels. The effective coupling is maximal: $\tau = 1$

Let us mention that appropriately modifying the argumentation presented in [18] one can show that the powerlaw tail M/y^2 turns out to be dictated by classical processes of exponential escape typical for fully chaotic systems [1].

The best candidates for checking the applicability of eq.(8) to real physical systems are realistic models of ballistic mesoscopic devices subject to an applied magnetic field that serves to break the TRS [8]. It is however quite clear that all the basic qualitative features of the distribution eq.(8) (in particular, the powerlaw behaviour $\rho(y) \propto My^{-2}$ for the overlapping resonance regime) should be valid for the systems with preserved TRS as well. Recent numerical data [19] support the validity of this conjecture.

Authors are very obliged to P.Seba for providing them with his unpublished numerical data [19] and to B.Khoruzhenko, N.Lehmann, A.Mirlin, V.Sokolov and H.-J.Stöckmann for illuminating discussions. Y.V.F. is grateful to D.Shepelyansky for attracting his attention to the powerlaw distribution found in [18].

The financial support by SFB 237 "Unordnung und grosse Fluctuationen" is acknowledged with thanks.

---

1. P.Gaspard in: *Quantum Chaos, Proc. of E. Fermi Summer School*, Eds. by G.Casati et.al., 1991, p.307.
2. U.Smilansky, in *Chaos and Quantum Physics, Proc. of the Les-Houches Summer School*, Session LII, Eds. by M.J.Giannoni et al., 1991, p.372.
3. O.Bohigas, in *Chaos and Quantum Physics*, see Ref. [2] , p.91.
4. J.Burgdörfer et al., Chaos, Solitons and Fractals **5**, 1235 (1995).
5. F.Haake et.al., Z. Phys. B **88**, 359 (1992); N.Lehmann et al., Nucl. Phys. A **582**, 223 (1995).
6. V.V.Sokolov and V.G.Zelevinsky, Nucl.Phys.A **504**, 562 (1989); Ann. Phys. (NY) **216**, 323 (1992).
7. E.Sobeslavsky, F.M.Dittes, and I.Rotter, J. Phys. A **28**, 2963 (1995).
8. R.A.Jalabert, A.D.Stone, and Y.Alhassid, Phys. Rev. Lett. **68**, 3468 (1992).
9. E.R.Mucciolo, V.N.Prigodin, and B.L.Altshuler, Phys. Rev. B **51**, 1714 (1995).
10. B.Gremaud, D.Delande, and J.C.Gay, Phys. Rev. Lett. **70**, 1615 (1993); K.Dupret, J.Zakrzewski, and D.Delande, Europh. Lett. **31**, 251 (1995).
11. H.Alt et al., Phys. Rev. Lett. **74**, 62 (1995); Nucl. Phys. A **560**, 293 (1993).
12. R.Schinke et al., J. Phys. B **28**, 2928 (1995).
13. J.J.M.Verbaarschot, H.A.Weidenmüller, and M.R.Zirnbauer, Phys. Rep. **129**, 367 (1985).
14. C.H.Lewenkopf and H.A.Weidenmüller, Ann. Phys. **212**, 53(1991).
15. O.Bohigas, in *Chaos and Quantum Physics*, see Ref. [2], p.91.
16. K.B.Efetov, Adv. Phys. **32**, 53 (1983).
17. P.A.Moldauer, Phys. Rev. **157**, 907 (1967); M.Simonius, Phys. Lett. **52B**, 279 (1974).
18. F.Borgonovi, I.Guarnery, and D.L.Shepelyansky, Phys. Rev. A **43**, 4517 (1991).
19. P.Seba, unpublished.