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EFFECTIVE QUARK LAGRANGIAN IN THE INSTANTON VACUUM WITH NONZERO MODES INCLUDED

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A new approach to effective theory of quarks in the instanton vacuum is presented. Exact equations for the quark propagator and Lagrangian are derived which contain contributions of all quark modes with known coefficients. The resulting effective Lagrangian differs from the standard one and resembles that of the Nambu–Jona-Lasinio model.

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1. The recent lattice data [1] provided evidence that instantons may be responsible for nonperturbative behavior of $q\bar{q}$ correlators [2] which makes the study of the quark dynamics in the instantonic vacuum [3–5] an important and fundamental problem.

To date practically all papers on the subject have relied upon the use of the so-called zero-mode approximation (ZMA) which amounts to comprising only zero quark mode in a single-instanton fermion propagator [3]. Correspondingly an ansatz for the partition function and effective quark Lagrangian (EQL) containing zero modes only have been proposed [5] and widely used in literature [6,7].

The purpose of this letter is to give a complete normal mode expansion of the EQL and of the quark propagator. Keeping only zero-mode coefficients in this expansion one retrieves the ansatz [5] for EQL. Naively one would expect that this choice of the coefficients yields dominant contribution to physical quantities, and thus justify ZMA. However the exact calculation of EQL presented below does not show this dominance. More intricate is the analysis of the quark propagator S in the instanton-anti-instanton vacuum. Here the zero-mode term survives but higher modes enter with coefficients of the same order. The similar feature can be seen in the quark partition function $<\det S^{-1}>$, where the average $<\dots>$ is defined below. Thus the new quark dynamics associated with nonzero modes emerges. The main features of this dynamics are outlined in what follows.

2. To make discussion transparent consider an ideal instanton gas with the superposition ansatz [8–10] and with zero net topological charge i.e. equal number of instantons and anti-instantons, $N_+ = N_- = N/2$:

$$A_\mu(x) = \sum_{i=1}^{N} A_\mu^{(i)}(x - R_i), \tag{1}$$

$$gA_\mu^{(i)} = \frac{\bar{\eta}_{a\mu\nu}(x - R_i)_\nu \rho^2 \Omega_i^+ \tau_a \Omega_i}{(x - R_i)^2[(x - R_i)^2 + \rho^2]}, \tag{2}$$

where Ω_i, R_i, and ρ are color orientation, position and scale-size of the i-th instanton.

The EQL is obtained from the Euclidean partition function after averaging over $\{\Omega_i, R_i\}$:

$$Z = \int D\psi D\psi^+ e^{-\int dx\psi^+ S^{-1}\psi} \prod_{i=1}^{N} \frac{dR_i}{V} d\Omega_i = \int D\psi D\psi^+ e^{-L_{eff}}, \tag{3}$$

where definitions here and in what follows are:

$$\begin{aligned} S_0^{-1} &= -i\hat{\partial} - im_f, \\ S_i^{-1} &= -i\hat{\partial} - g\hat{A}^{(i)} - im_f, \\ S^{-1} &= -i\hat{\partial} - g\hat{A} - im_f. \end{aligned} \tag{4}$$

Next we introduce the standard set of eigenfunctions $\{u_n^i\}$, $n = 0, 1, 2, \ldots$,

$$(-i\hat{\partial} - g\hat{A}^{(i)})|u_n^i> = \lambda_n |u_n^i> . \tag{5}$$

Then S^{-1} given by (4) has a formal representation as a sum over normal modes

$$S^{-1} = S_0^{-1} + \sum_{i,m,n} S_0^{-1}|u_m^i> \varepsilon_{mn}^i < u_n^i|S_0^{-1}, \tag{6}$$

where $\hat{\varepsilon}$ can be represented either as

$$\varepsilon_{mn}^i = - < u_m^i|(S_i - S_0)[1 + S_0^{-1}(S_i - S_0)]^{-1}|u_n^i >, \tag{7}$$

or simply as

$$\varepsilon_{mn}^i = -g < u_m^i|S_0\hat{A}^{(i)}S_0|u_n^i > . \tag{8}$$

Performing in (3) averaging with the help of cumulant or cluster expansion, one obtains L_{eff} in the form

$$L_{eff} = \int dx\psi^+ S_0^{-1}\psi + \sum_{n=2}^{\infty}(-1)^{n-1}\left(\frac{2V}{N}\right)^{n-1} \sum_{fmm'} \int d\Gamma_n \det{}_{k,l}^{(n)} J_{kl}, \tag{9}$$

where

$$d\Gamma_n = \prod_{j=1}^{n} \frac{dp_j}{(2\pi)^4}\frac{dp'_j}{(2\pi)^4}(2\pi)^4\delta\left(\sum_j (p_j - p'_j)\right), \tag{10}$$

$$J_{kl} = \left(\psi^{f_k}(p_k)\right)^+ M_{m_k m'_l}^{f_k f_l}(p_k, p'_l)\psi^{f_l}(p'_l), \tag{11}$$

and similarly to [5,6] we have introduced the vertices

$$M_{mm'}^{gr}(p, p') = \frac{N}{2VN_c}(\hat{p} - im_g)\varphi_m(p)\varepsilon_{mm'}^i\varphi_{m'}^+(p')(\hat{p}' - im_r), \tag{12}$$

with $\varphi_m(p)$ being the form factor of u_m^i in momentum space.

Summation in (9) starts from $n = 2$ since the $n = 1$ term drops out as a result of integration over color orientations.

The EQL in (9) is a sum of $n \times n$ determinants. If one confines oneself to ZMA i.e. puts ε^i_{00} finite and $\varepsilon^i_{m>0,n>0}$ equal to zero the sum runs only over $n \leq N_f$. This restriction is due to Grassmann nature of J_{kl}. Thus even in ZMA one obtains e.g. for $N_f = 3$ three 2×2 determinants and one 3×3 determinant. Only the last one is present in the ansatz [5] with the identification $\varepsilon^i_{00} \equiv \varepsilon$, $M_{00} \equiv M$. Therefore our results are in contrast to the common lore according to which for a given number of flavors N_f the only vertex appearing in the chiral limit contains $2N_f$ quark operators. We can reproduce this result for $N_f = 2$ if only ε^i_{00} is kept nonzero, while for $N_f = 3$ this conjecture does not suffice and we get additional $4q$ terms.

Consider now ε^i_{00} using (8). In the chiral limit the operator $S_0 \hat{A}^{(i)} S_0$ is chirally odd while instanton zero-mode has definite chirality and therefore ε^i_{00} vanishes, for $m_f \neq 0$ one has

$$\varepsilon^i_{00} = O(m_f), \qquad m_f \to 0. \tag{13}$$

At the same time nonzero modes u^i_{mn} do not have definite chirality and hence matrix elements ε^i_{mn} do not vanish as $m_f \to 0$. Thus ZMA in the naive sense of dominance of zero-mode terms in the EQL is not supported by our calculations. In the next section we discuss what it means in terms of quark propagator.

3. Now we turn to the quark propagator, expressing it again through ε^i_{mn}. Inverting (6) one finds

$$S = S_0 - \sum_{ijmn} |u^i_m> \left(\frac{1}{\hat{\varepsilon}^{-1} + \hat{V}} \right)^{ij}_{mn} < u^j_n|, \tag{14}$$

where $(\hat{\varepsilon})^{ij}_{mn} = \delta_{ij} \varepsilon^i_{mn}$, and

$$(\hat{V})^{ij}_{mn} = < u^i_m | S_0^{-1} | u^j_n > . \tag{15}$$

Note that summation in (14) extends over different instantons and hence over u^i_0 and u^j_0 of different chiralities. Equation (14) has to be compared to the following expression common to most papers on the subject [3,5,6]

$$S = S_0 - \sum_{i,j} |u^i_0> \left(\frac{1}{2im + V} \right)^{ij}_{00} < u^j_0|, \tag{16}$$

which contains only zero modes contributions. To derive (16) one starts with the following approximation for the quark propagator in a single instanton field [3,5]:

$$S_i = (-i\hat{\partial})^{-1} + \frac{|u^i_0 >< u^i_0|}{-im}. \tag{17}$$

Introducing this ansatz into expression (7) for ε^i_{mn} we get

$$\varepsilon^i_{00} = \frac{1}{2im}, \qquad \varepsilon^i_{m>0,n>0} = 0. \tag{18}$$

Using this form of $\hat{\varepsilon}$ in (14) one recovers the standard ZMA (16). Now, comparing (18) to (13) we conclude that ansatz (17) is unjustified. Actually when ε^i_{00} vanishes in the chiral limit in line with (13), the propagator (14) still contains terms $|u_0^i><u_0^j|$, but with coefficients depending upon higher modes contributions V_{mn}^{ij}. This can be seen expanding (14) in series in powers of ε, i.e.

$$S = S_0 - \sum_{i,j,m,n} |u_m^i > (\hat{\varepsilon} - \hat{\varepsilon}\hat{V}\hat{\varepsilon} + \hat{\varepsilon}\hat{V}\hat{\varepsilon}\hat{V}\hat{\varepsilon} - \ldots)_{mn}^{ij} < u_n^j|. \tag{19}$$

If one neglects nonzero modes in V_{mn} in (19), then the coefficient of $|u_0^i><u_0^j|$ automatically vanishes. To make contact with popular instantonic technique [3,5,6] where only zero modes are kept in the quark wave functions of instantons I and anti-instantons $\bar{I}$, we rearrange the series for the quark propagator and partition function, using the relation $\hat{\varepsilon}\hat{V} = S_0\hat{A}$ and separate out the terms containing the overlap of $I\bar{I}$ zero-modes.

In the standard ZMA these terms are assumed to be dominant while the overlaps of nonzero modes are neglected. Our expression (19) includes both types of contributions and does not show zero-mode dominance. Therefore we propose to study the new EQL derived above and calculate physical quantities like chiral quark mass and chiral condensate in order to estimate the contribution of nonzero-modes.

It is worth noting that the consistency of the approximation (17) was questioned in [9] in connection to the calculation of the two–point correlation function. It was shown in [9] that it is absolutely necessary to keep the order $\sim m$ terms in S_i. However since for massive fermions the single instanton propagator S_i is not explicitly known the effects of higher modes and finite mass have been investigated only numerically [11].

Finally let us see the effect of nonzero modes in the quark partition function, which is obtained from (3) integrating first over quark fields. Using (6) for S^{-1} one easily obtains

$$Z/Z_0 = \prod_{f=1}^{N_f} \det(1 + \hat{\varepsilon}\hat{V}), \tag{20}$$

where $\hat{\varepsilon}$ and $\hat{V}$ are the same matrices as in (14), (15). We may now repeat arguments presented after (19) to demonstrate the presence of nonzero modes and the absence of zero-modes dominance.

4. One may wonder, why ZMA i.e. keeping only zero-modes in EQL might be invalid even though phenomenologically it looks like giving reasonable results [5-6, 12]. One of the reasons might be that $\varepsilon_{00}^i \equiv \varepsilon$ has been treated as a parameter connected to the properties of the instanton vacuum via the relation $\varepsilon \sim (N_c V/N\rho^2)^{1/2}$, while the properties of the vacuum have been in turn adjusted to the correct value of the gluon condensate.

Our results are at first sight in contradiction to the Banks–Casher relation [13] which connects the chiral condensate with the density of global (quasi) zero modes. The standard picture suggests that the later originate from individual zero modes, and hence would disappear as soon as (13) holds. However here the standard picture may fail. An insight into its possible failure is provided by quantum mechanics of collective levels in N potential wells in $4D$. If each of the well has one loosely bound level and continuum (equivalent to zero mode and nonzero modes), then the approximation of keeping only the bound state poles in the Green's functions of each well is known to give an inadequate description of collective bound states [14]. More than that, the pole approximation is a poor one even for the Green's function of the individual well, and instead the so-called unitary pole approximation has to be used [15].

5. To summarize, we have outlined the new approach to the effective theory of quarks in the instanton vacuum. Our EQL is similar to that of Nambu–Jona-Lasinio [16], namely it starts from $4q$ term which might play an important role in phenomenology. Analogy to NJL model calls for construction of gap equation yielding chiral quark mass and quark condensate. Also, the bosonization procedure has to be performed yielding the effective chiral Lagrangian for Nambu–Goldstone modes. Finally, the low density limit deserves a special discussion. This program is in progress now and will be reported elsewhere.

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