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FIELD THEORY OF MESOSCOPIC FLUCTUATIONS IN SUPERCONDUCTOR/NORMAL-METAL SYSTEMS

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Thermodynamic and transport properties of normal disordered conductors are strongly influenced by the proximity of a superconductor. A cooperation between mesoscopic coherence and Andreev scattering of particles from the superconductor generates new types of interference phenomena. We introduce a field theoretic approach capable of exploring both averaged properties *and mesoscopic fluctuations* of superconductor/normal-metal systems. As an example the method is applied to the study of the level statistics of a SNS-junction.

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Physical properties of both superconductors and mesoscopic normal metals are governed by mechanisms of macroscopic quantum coherence. Their interplay in hybrid systems comprised of a superconductor adjacent to a normal metal gives rise to qualitatively new phenomena (see Ref. [1] for a review): aspects of the superconducting characteristics are imparted on the behaviour of electrons in the normal region. This phenomenon, known as the "proximity effect", manifests itself in a) the *mean* (disorder averaged) properties of SN-systems being substantially different from those of normal metals and b) various types of mesoscopic *fluctuations* which not only tend to be larger than in the pure N-case but also can be of qualitatively different physical origin. Although powerful quasi-classical methods, based largely on the pioneering work of Eilenberger [2] and Usadel [3], have been developed to analyse the manifestations of the proximity effect in average characteristics of SN-systems, far less is known about the physics of mesoscopic fluctuations: while the quasi-classical approach is not tailored to an analysis of fluctuations, standard diagrammatic techniques [4] used in the study of N-mesoscopic fluctuations can often *not* be applied due to the essentially non-perturbative influence of the fully established proximity effect. Important progress was made recently by extending the scattering formulation of transport in N-mesoscopic systems to the SN-case [1]. This approach has proven powerful in the study of various transport fluctuation phenomena but is not applicable to the study of fluctuations on a local and truly microscopic level.

In the present Letter we introduce a general framework that combines key elements of the quasi-classical approach with more recent methods developed in N-mesoscopic physics into a unified approach. As a result we obtain a formalism that can be applied to the general analysis of mesoscopic fluctuations superimposed on a proximity effect influenced mean background. In order to demonstrate the practical use of the approach we will consider the example of *spectral fluctuations* as a typical representative of a mesoscopic phenomenon. The density of states (DoS) of N-mesoscopic systems exhibits quantum fluctuations around its disorder averaged mean value which can be described in terms of various types of universal statistics. The analogous question in the SN-case – what types of statistics govern the disorder induced fluctuation behaviour of the *proximity effect* *influenced* DoS? – has not been answered so far. Our main result, the emergence of some kind of modified Wigner Dyson statistics [5], will be derived below.

To be specific we consider the geometry of a quasi-1D SNS junction, where the N-region is of length L and the complex order parameters of the adjacent S-regions differ in phase by φ. It is well known [6] that even the mean DoS of the SNS-system exhibits non-trivial behavior which is difficult to describe within standard perturbative schemes: states which fall within the superconducting gap, Δ, are confined to the normal metal. The proximity effect then further induces a minigap in the DoS of the *normal* region around the Fermi-energy, ϵ_F, whose size of $O(E_c = D_n/L^2)$ depends sensitively on φ (D_n is the diffusion constant and $\hbar = 1$ throughout). To analyse the fluctuation behavior of the DoS, $\nu(\epsilon)$, around its disorder averaged background, $\langle \nu(\epsilon) \rangle$, we will consider the two-point correlation function [7], $R_2(\epsilon, \omega) = \langle \nu(\epsilon) \rangle^{-2} \langle \nu(\epsilon + \omega/2) \nu(\epsilon - \omega/2) \rangle_c$. The starting point of our analysis is the Gor'kov equation for the matrix advanced/retarded (a/r) Green function [2]

$$\mathcal{G}_\epsilon^{r,a} = \begin{pmatrix} G_\epsilon^{r,a} & F_\epsilon^{r,a} \\ F^{\dagger}{}_\epsilon^{r,a} & G^{\dagger}{}_\epsilon^{r,a} \end{pmatrix}, \tag{1}$$

where

$$\left[\epsilon_F - \frac{1}{2m}\left(\hat{p} - \frac{e}{c}\mathbf{A}\sigma_3^{ph}\right)^2 - V(\mathbf{r}) + \left(\hat{\Delta}(\mathbf{r}) + \epsilon_\pm\right)\sigma_3^{ph}\right] \mathcal{G}_\epsilon^{r,a}(\mathbf{r}, \mathbf{r}') = \delta^d(\mathbf{r} - \mathbf{r}'), \tag{2}$$

$\epsilon_\pm \equiv \epsilon \pm i0$, $\mathbf{A}$ is the vector potential of an external magnetic field, $\hat{\Delta} = \Delta\sigma_1^{ph} \exp(-i\varphi\sigma_3^{ph})$ represents the (spatially dependent) complex order parameter with phase φ, and Pauli matrices $\vec{\sigma}^{ph}$ operate in the Nambu or particle/hole (ph) space. The impurity potential in the N-region is taken to be Gaussian δ-correlated with zero mean and correlation $\langle V(\mathbf{r})V(\mathbf{r}') \rangle = \delta^d(\mathbf{r} - \mathbf{r}')/2\pi\nu\tau$, where ν denotes the DoS of the bulk normal metal at ϵ_F, and τ represents the mean free scattering time. In the following the complex order parameter in the S-region is *imposed* and not obtained self-consistently [8]. Where the S- and N-region are distinct (as in the SNS junction), the bulk DoS, $\nu_{n,s}$ and scattering time, $\tau_{n,s}$ will be chosen independently.

Traditionally the *impurity averaged* Green function (1) is computed within a quasi-classical approximation, i.e. the Schrödinger equation (2) is reduced to an effective transport equation, the Eilenberger equation [2], which in the dirty limit simplifies further to the diffusive Usadel equation [3]. Here we develop a field theoretical formulation that integrates concepts of the quasi-classical formalism into a more general framework allowing for the computation of disorder averaged *products of* Green functions, a necessary requirement for the calculation of correlation functions such as R_2. The basic strategy will be to start from a (microscopically derived) generating functional whose points of stationary phase obey the Usadel equation. By investigating fluctuations around this quasi-classical limit, correlations between the different Green functions will be explored. In the following we formulate this program in more detail.

As in the pure N-case, ensemble averaged products of advanced and retarded Gorkov Green functions can be described in terms of generating functionals of nonlinear σ-model type [9] (see Ref. [10] for a review on the σ-model analysis of Green functions in N-mesoscopic physics). In the dirty limit, $(\epsilon, \Delta) < \tau^{-1} \ll \epsilon_F$, the generalization of the supersymmetric N-type σ-model [10] reads

$$\int_{Q^2=} DQ(\cdots)e^{-S[Q]}, \qquad (3)$$

$$S[Q] = -\frac{\pi\nu}{8}\int \mathrm{str}\Big[D(\widetilde{\partial}Q)^2 + 4iQ(\widetilde{\Delta} + \epsilon + \frac{\omega_+}{2}\sigma_3^{ar})\sigma_3^{ph}\Big],$$

where $\widetilde{\partial} = \partial - i(e/c)[\mathbf{A}_\phi\sigma_3^{tr}\otimes\sigma_3^{ph},\cdot]$ represents a covariant derivative, $\mathbf{A}_\phi = \mathbf{A} + c/(2e)\partial\phi$ accounts for both the external field and the phase of the order parameter, $\widetilde{\Delta} = \Delta\sigma_2^{ph}$, the Pauli matrices $\vec{\sigma}^{fb}$, $\vec{\sigma}^{tr}$ and $\vec{\sigma}^{ar}$ operate in fermion/boson, time-reversal and ar-blocks respectively [10]. The symbol D stands for a space dependent diffusion constant which may take separate values, denoted as $D_{n,s}$, in the N- and S-regions. Although specific pre-exponential source terms (denoted by ellipses in Eq.(3)) must be chosen according to any given correlation function (such as R_2), their precise form does not influence the analysis below and we therefore refer to Ref. [10] for their detailed structure. The integration in (3) extends over a 16×16-dimensional matrix field $Q = T^{-1}\sigma_3^{ph}\otimes\sigma_3^{ar}T$, whose symmetries are identical with those of the conventional σ-model [10].

The expression (3) differs in two respects from the σ-model for N-systems: i) the appearance of a ph-space associated with the 2×2-matrix structure of the Gorkov Green function, and ii) the presence of the order parameter $\tilde{\Delta}$. Whereas i) can be accounted for by a doubling of the matrix dimension of the field Q, ii) calls for more substantial modifications: for $\Delta \neq 0$ standard perturbative schemes for the evaluation of the functional (3) fail [11], an indication of the fact that the superconductor influences the properties of the normal metal heavily. Under these conditions a more efficient approach is first to subject the action to a mean field analysis and then to consider fluctuations around a newly defined – and generally space dependent – stationary field configuration. A variation of the action (3) with respect to Q, subject to the constraint $Q^2 =$, generates a non-linear equation for the saddle-point,

$$D\widetilde{\partial}_i(\bar{Q}\widetilde{\partial}_i\bar{Q}) + \Big[\bar{Q}, \Delta\sigma_1^{ph} - i(\epsilon + \omega_+\sigma_3^{ar})\sigma_3^{ph}\Big] = 0. \qquad (4)$$

Current conservation implies the boundary condition [12], $\sigma_n Q\partial_x|_{x_n}Q = \sigma_s Q\partial_x|_{x_s}Q$, where $\sigma_{n,s} = e^2\nu_{n,s}D_{n,s}$ denotes the conductivity and $\partial_x|_{x_{n(s)}}$ is a normal derivative at the N(S)-side of the interface.

An inspection of (4) shows that only the particle/hole components of the matrix field $\bar{Q}$ are coupled by the saddle point equation. It is thus sensible to make a block diagonal ansatz $\bar{Q} = \mathrm{bdiag}(q_+, q_-)$, where the eight dimensional retarded, q_+ and advanced, q_- subblocks are diagonal in both time reversal and boson/fermion space. Noting that the saddle point configuration $-i\pi\nu q_\pm$ of the nonlinear σ-model is associated with the impurity averaged retarded/advanced Green function [10], we identify Eq. (4) as the Usadel equation. The general connection between the σ-model formalism and the quasi-classical approach has first been noticed in [13].

Eq. (4), in its interpretation as the Usadel equation, has been discussed at length in the literature [6, 14, 15]. Although complex in general, the solutions have a simple qualitative geometric interpretation: employing the explicit parametrization $q_\pm = \mathbf{q}_\pm\cdot\vec{\sigma}^{ph}$, eq. (4) describes the gradual rotation of the three dimensional vector $\mathbf{q}$ from a direction almost parallel to $\hat{e}_1$ in the bulk superconductor to a value aligned with $\hat{e}_3$ deep in the normal metal.

So far our analysis has been for SN-systems of a general geometry. Specializing the discussion to the SNS-junction, we set $\Delta(\mathbf{r}) \equiv \Delta\Theta(|x| - L/2)$ constant inside the superconductor ($\Delta \gg E_c$), and zero in the normal region, with a phase $\pi/2 + \mathrm{sgn}(x)\varphi/2$. The saddle-point equation depends sensitively on both the presence or absence of an external magnetic field and the phase difference between the order parameters. Taking the external field to be zero, it is convenient to focus on two extreme cases: (i) $\varphi = 0$ (orthogonal symmetry), and (ii) $\varphi \gg 1/\sqrt{g}$ (unitary symmetry). Here $g = E_c/\bar{d} \gg 1$ denotes the dimensionless conductance and $\bar{d}$ represents the bulk single-particle level spacing of the normal metal.

The *disorder averaged local DoS* can be obtained from the analytical solution of the Usadel-saddle point equation [6, 14 – 17] as $\nu(\mathbf{r}) = \nu\mathrm{Re}[q_+(\mathbf{r})]_3$. The most striking feature of the average DoS is the appearance of a spatially constant minigap in the N-region. The gap attains its maximum width E_c at $\varphi = 0$ and shrinks to 0 as φ approaches π [15].

We next turn to the main subject of this Letter, the issue of *fluctuations around the Usadel saddle-point*. Employing the parametrization $Q = T^{-1}\bar{Q}T$, $T \neq$, three qualitatively different types of fluctuations can be identified: (a) fields T which are diagonal in the space of advanced and retarded components, (b) T's which commute with all matrices σ_i^{ph} but mix advanced and retarded components, and (c) T's fulfilling neither of the conditions (a) and (b). Fluctuations of (a)-type preserve the ar-diagonal structure of the saddle point. These fluctuations do not give rise to correlations between advanced and retarded Green functions. Nonetheless, they are of physical significance: Quantum corrections to the Usadel solution, most importantly the renormalization of the minigap by weak localization effects and the existence of rare prelocalized states [13, 18] below the gap, are described by fluctuations of this type. We postpone further discussion of these results to a separate paper [17] and, instead, turn to the discussion of the second type of fluctuations, (b).

Consider the saddle point equation (4) in the simple case $\omega = \varphi = 0$. Obviously, as it commutes with all matrices σ_i^{ph}, any spatially constant rotation T of type (b) gives rise to another solution. In other words, the (b)-fluctuations represent Goldstone modes with an action that vanishes in the limit of spatial constancy and $\omega \to 0$. Since any T diagonal in ph-space inevitably has to couple between advanced and retarded indices [10], these modes lead to *correlations between advanced and retarded Green functions* (and thereby to mesoscopic fluctuations) which become progressively more pronounced as ω approaches zero.

In the limit of small frequencies $\omega < E_c$, the *ergodic regime*, the global zero mode $Q_0 = T_0^{-1}qT_0$, $[T_0, \vec{\sigma}^{ph}] = 0$, $T_0(\mathbf{r}) = \mathrm{const}$, plays a unique role: whereas fluctuations with non-vanishing spatial dependence give rise to contributions to the action of $O(g \gg 1)$ [10], this mode couples *only* to the frequency difference ω. Restricting attention to the pure zero mode contribution, we obtain the effective action

$$S_0[Q_0] = -i\frac{\pi}{2}\frac{\omega_+}{\bar{d}(\epsilon)}\mathrm{str}\,[Q_0\sigma_3^{ar}], \tag{5}$$

where $\bar{d}(\epsilon) = (\int \nu(\epsilon))^{-1}$ denotes the average level spacing and the ph-degrees of freedom have been traced out. From this result it follows [10] that, in the ergodic regime, the spectral statistics of an SNS system is governed by Wigner – Dyson fluctuations [5, 19] of (i) orthogonal or (ii) unitary symmetry superimposed upon an energetically non-uniform mean DoS. Furthermore, a comparison of Eq. (5) with the analogous action for N-systems [10] shows the correlations to depend on an average level spacing that is effectively *halved*. This reflects the strong "hybridization" of levels at energies $\sim \epsilon_F \pm \epsilon$ induced by Andreev scattering at the SN-interface.

In further contrast to N-systems, the range over which Wigner – Dyson statistics apply turns out to be greatly diminished by non-universal fluctuations [20, 21]. This is a consequence of the presence of fluctuations of type (c), coupling between advanced/retarded *and* particle/hole components. The detailed analysis of the (c)-type fluctuations is cumbersome and will be deferred to a forthcoming publication [17]. Here we only report that a perturbative integration over these modes leads to an exponential suppression of the DoS–fluctuations already for energy separations $\omega/\bar{d}(\epsilon) \sim \sqrt{g}$. This is in contrast to the pure N-case where the Wigner – Dyson regime (prevailing up to frequencies $\omega \simeq E_c$) is succeeded by other forms of *algebraically* decaying spectral statistics in the high frequency domain $\omega > E_c$ [22].

In conclusion a general framework has been developed in which the interplay of mesoscopic quantum coherence phenomena and the proximity effect can be explored. An investigation of the spectral statistics of an SNS geometry revealed that level correlations are Wigner – Dyson distributed with strong non-universal corrections at large energy scales. Finally, we remark that for quantum structures in which transport is not diffusive but ballistic and boundary scattering is irregular, a ballistic σ-model involving the classical Poisson bracket can be derived [23]. In this case, the saddle-point condition recovers the Eilenberger equation of transport [2].

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