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A SIMPLE APPROACH TO THE HALL CONDUCTIVITY EVALUATION IN IMPURE METALS WITHIN THE GREEN FUNCTION FORMALISM

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A simple method to deal with the Hall effect in metals with short-ranged impurities in a weak magnetic field is proposed. The method is based on a Schwinger representation for the electron Green function in the magnetic field. The method efficiency is demonstrated on an calculation of the antisymmetric components of the conductivity tensor at finite wave vector.

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Despite the discovery of the quantum Hall effect, theoretical analysis of the Hall effect in metals in *weak* magnetic fields has been attracting a lot of attention [1-6] because of its practical as well as scientific interest. Quantum-mechanical treatments of the issue published up to now are based on a method proposed in [2]. In this method, an electron system subjected to external uniform magnetic field is considered to be a limiting case of the system placed in a fictitious nonuniform magnetic field with the vector potential $\mathbf{A}(\mathbf{r}) = \mathbf{A}(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}}$. The wavevector $\mathbf{q}$ is later tended to zero, $\mathbf{q} \to 0$, to recover the case of uniform magnetic field. The nonuniform field results in inhomogeneity of the system which induces the carrier diffusion. In the Feynman-diagram language, this fact means the appearance of the diffusion poles. The necessity of eliminating the poles to obtain a divergent-free expression for the Hall conductivity $\sigma_{ij}^{(H)}$ makes the method somewhat cumbersome (in spite of some improvements published later [5]). Furthermore, the method has only been formulated for evaluation of the Hall conductivity at zero wave vector.

The purpose of the present paper is to propose another quantum approach which is convenient in the particular case of p-independent impurity scattering. The distinctive feature of our approach is that the external magnetic field is considered to be uniform from the very beginning. Therefore, no carrier diffusion takes place and we only have the Feynman diagrams that do not contain the diffusion poles. In addition, the method can be extended to finite wave vectors without any difficulties, providing an expression for $\sigma_{ij}^{(H)}(\omega, \mathbf{q})$. (Here we consider only macroscopic systems.)

Let $\mathbf{a}(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x})$ be the vector potentials of the driving electric field $\mathbf{E} = -\frac{\partial \mathbf{a}}{c\partial t}$ and external constant magnetic field $\mathbf{H} = \nabla \times \mathbf{A}$, respectively. The linear response to $\mathbf{a}$ is known to be determined by the current-current correlation function

$$Q_{ij}(\mathbf{x}, \mathbf{y}; i\omega_n) = \int_0^{1/T} d\tau < \hat{T}\hat{J}_i(\mathbf{x}, \tau)\hat{J}_j(\mathbf{x}, 0) >, \tag{1}$$

where T in the upper limit of the integral is the temperature and $\hat{T}$ in the angular brackets is the time-ordering operator. In what follows we assume that the so-called "diamagnetic

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part" of the response, $(ne^2/m)\delta(\mathbf{x}-\mathbf{y})$, is canceled by the appropriate part of Q_{ij} (analytically continued to the real frequency axis) in the usual manner. The electron charge is considered to be $-e$. Then the current operator $\hat{\mathbf{J}}$ is the sum of the kinetic part,

$$\hat{\mathbf{j}}_{kin} = -\frac{ie}{2m}[\psi^+(\mathbf{r})\nabla\psi(\mathbf{r}) - \nabla\psi^+(\mathbf{r})\psi(\mathbf{r})],$$

and the diamagnetic part,

$$\hat{\mathbf{J}}_{dia} = -\frac{e^2}{mc}\mathbf{A}(\mathbf{r})\psi^+(\mathbf{r})\psi(\mathbf{r}).$$

For the sake of brevity, we drop the spinor indices at the field operators and Green functions. The field $\mathbf{H}$ is assumed to be small so that $\omega_c\tau < 1$ (here $\omega_c = eH/mc$ is the cyclotron frequency and τ is the mean free time) and, hence, one can neglect the Landau quantization and use the representation for the electron Green function originally due to Schwinger [7]

$$G_\epsilon(\mathbf{r},\mathbf{r}';\mathbf{H}) = \exp\left(\frac{ie}{c}\int_{\mathbf{r}}^{\mathbf{r}'} d\mathbf{r}''\cdot\mathbf{A}(\mathbf{r}'')\right) G_\epsilon(\mathbf{r}-\mathbf{r}';0), \tag{2}$$

where $G_\epsilon(\mathbf{r}-\mathbf{r}';0)$ is the Green function in zero magnetic field. In this paper, the symmetric gauge, $\mathbf{A} = \frac{1}{2}\mathbf{H}\times\mathbf{r}$, is chosen and $\hbar = 1$ is set throughout. It should be noticed that such a representation has been successfully employed in many branches of physics, e.g. in superconductivity [8] and plasma physics [9], but never (to our knowledge) in the theory of Hall effect in weak magnetic fields. The current operator can be written as

$$\hat{\mathbf{J}}(\mathbf{x}) = \lim_{\mathbf{x}'\to\mathbf{x}} \hat{\mathbf{J}}(\mathbf{x},\mathbf{x}'),\ \hat{\mathbf{J}}(\mathbf{x},\mathbf{x}') = \psi^+(\mathbf{x})\mathbf{J}(\mathbf{x},\mathbf{x}')\psi(\mathbf{x}'), \tag{3}$$

$$\mathbf{J}(\mathbf{x},\mathbf{x}') = -e\left[\frac{\nabla_x - \nabla_{x'}}{2im} + \frac{e}{2mc}\mathbf{A}(\mathbf{x}+\mathbf{x}')\right]. \tag{4}$$

The thermal average in (1) gives rise to a set of diagrams [10] which have the form of a two-vertex electron loop with various impurity-line insertions and with the current operator (in the Schrödinger representation) at both these vertices. The impurity potential $U(\mathbf{r})$ will be assumed short-ranged so that $< U(\mathbf{r})U(\mathbf{r}') >\sim \delta(\mathbf{r}-\mathbf{r}')$. If, as usually, one includes the impurity self-energy into the electron Green function, the remaining impurity-lines necessarily connect the upper and lower electron lines forming a given loop and can be considered as impurity vertex corrections. Without an external magnetic field, the evaluation of such diagrams is greatly simplified by making use of the Fourier transform. With the magnetic field present, the translational invariance of the Green function is broken due to the Schwinger's factors

$$\Phi(\mathbf{x},\mathbf{y}) = \exp\left(\frac{ie}{c}\int_{\mathbf{x}}^{\mathbf{y}} d\mathbf{r}\cdot\mathbf{A}(\mathbf{r})\right) = \exp\left(\frac{ie}{2c}\mathbf{H}\cdot(\mathbf{x}\times\mathbf{y})\right), \tag{5}$$

making the immediate application of the Fourier method impossible. However, the general theorem [11], that any electron loop taken as a whole has to be invariant under translations, means that the translational invariance must be recovered by explicit evaluation of the loop. Let us show how this happens in the case of "empty" loop (without impurity insertions). The analytical expression for the loop contains

$$\lim_{\substack{\mathbf{x}'\to\mathbf{x}\\ \mathbf{y}'\to\mathbf{y}}} J_i(\mathbf{x},\mathbf{x}')J_j(\mathbf{y},\mathbf{y}')\Phi(\mathbf{x},\mathbf{y}')\Phi(\mathbf{y},\mathbf{x}')G(\mathbf{x}-\mathbf{y}')_{i\epsilon+i\omega}G(\mathbf{y}-\mathbf{x}')_{i\epsilon}, \tag{6}$$

which, in view of the relations

$$\frac{\nabla_x}{2i}\Phi(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{x},\mathbf{y})\left[\frac{\nabla_x}{2i} - \frac{e}{2c}\mathbf{A}(\mathbf{y})\right], \quad \frac{\nabla_y}{2i}\Phi(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{x},\mathbf{y})\left[\frac{\nabla_y}{2i} + \frac{e}{2c}\mathbf{A}(\mathbf{x})\right] \tag{7}$$

can be recast as

$$\lim_{\substack{\mathbf{x}'\to\mathbf{x}\\ \mathbf{y}'\to\mathbf{y}}} \Phi(\mathbf{x},\mathbf{y}')\Phi(\mathbf{y},\mathbf{x}')\left[\frac{\nabla_x-\nabla_{x'}}{2im} + \frac{e}{2mc}\mathbf{A}(\mathbf{x}+\mathbf{x}'-\mathbf{y}-\mathbf{y}')\right]_i \times$$

$$\times\left[\frac{\nabla_y-\nabla_{y'}}{2im} - \frac{e}{2mc}\mathbf{A}(\mathbf{x}+\mathbf{x}'-\mathbf{y}-\mathbf{y}')\right]_j G(\mathbf{x}-\mathbf{y}')_{i\epsilon+i\omega}G(\mathbf{y}-\mathbf{x}')_{i\epsilon}. \tag{8}$$

In (8), the derivatives do not act on the phase factors; therefore, one can set in *the factors* $\mathbf{y}'=\mathbf{y}$ and $\mathbf{x}'=\mathbf{x}$. As a result, the product of the factors reduces to unity. Now one should substitute into (8) the Fourier representation for the free electron Green functions and perform the coordinate differentiations. After that one can already set $\mathbf{y}'=\mathbf{y}$ and $\mathbf{x}'=\mathbf{x}$ in remaining functions. Then the total expression for the loop takes the explicitly translational-invariant form

$$-e^2\sum_{\epsilon,\mathbf{p},\mathbf{q}_1}\left[\frac{\mathbf{p}}{m} + \frac{e}{mc}\mathbf{A}(\mathbf{x}-\mathbf{y})\right]_i e^{i\mathbf{q}_1\cdot(\mathbf{x}-\mathbf{y})}\times$$

$$\times\, G_{i\epsilon+i\omega}\left(\mathbf{p}+\frac{\mathbf{q}_1}{2}\right)G_{i\epsilon}\left(\mathbf{p}-\frac{\mathbf{q}_1}{2}\right)\left[\frac{\mathbf{p}}{m} - \frac{e}{mc}\mathbf{A}(\mathbf{x}-\mathbf{y})\right]_j. \tag{9}$$

The part of the response $Q^R_{ij}(\omega,\mathbf{q}) = \int d^3(\mathbf{x}-\mathbf{y})e^{-i\mathbf{q}_1\cdot(\mathbf{x}-\mathbf{y})}Q_{ij}(\mathbf{x}-\mathbf{y};\omega+i0)$ linear in $\mathbf{H}$ can be written [with the help of the identity $\int d^3r e^{i\mathbf{q}\cdot\mathbf{r}}\mathbf{r} = -i(2\pi)^3\nabla_{\mathbf{q}}\delta(\mathbf{q})$] as

$$\sigma^{(H)}_{ij}(\omega,\mathbf{q}) = \frac{e^2}{\pi}\left(\frac{ie}{4mc}\right)\left[(\mathbf{H}\times\nabla^{R-A}_{\mathbf{p}})_i G^R_\omega\left(\mathbf{p}+\frac{\mathbf{q}}{2}\right)G^A_0\left(\mathbf{p}-\frac{\mathbf{q}}{2}\right)\frac{p_j}{m} -\right.$$

$$\left. -\frac{p_i}{m}G^R_\omega\left(\mathbf{p}+\frac{\mathbf{q}}{2}\right)G^A_0\left(\mathbf{p}-\frac{\mathbf{q}}{2}\right)(\mathbf{H}\times\nabla^{R-A}_{\mathbf{p}})_j\right], \tag{10}$$

where the superscript $R(A)$ stands for the retarded (advanced) part of the function and the operator $\nabla^{R-A}_{\mathbf{p}}$ in the first term acts only on the pair of the Green functions according to the rule $\nabla^{R-A}_{\mathbf{p}}G^RG^A = (\nabla_{\mathbf{p}}G^R)G^A - G^R(\nabla_{\mathbf{p}}G^A)$, but *not* on the velocity-vertex $\mathbf{p}/m$. The operator $\nabla^{R-A}_{\mathbf{p}}$ in the second term acts to the left in the same way. Proceeding in the same manner, one can see that the contribution of all impurity-ladder diagrams has the form depicted in Figure. The evaluation of $\sigma^{(H)}_{ij}$ is now reduced to the level of, say, the Drude conductivity. It should be stressed that the δ-functional form of the impurity-field correlator is essential in deriving this result. Straightforward calculations yield

$$\sigma^{(H)}_{ij}(\omega,\mathbf{q}) = \sigma_3\frac{3\omega_c\tau}{2(1-i\Omega)^2}e_{ijn}\left\{h_n\left(\frac{\beta-\gamma}{t^2}\right)\frac{i\Omega}{1-\beta-i\Omega}+\right.$$

$$+\hat{q}_n(\hat{\mathbf{q}}\cdot\mathbf{h})\left[\left(\frac{\gamma-\beta}{t^2}\right)\frac{i\Omega}{1-\beta-i\Omega}-2\frac{1-\beta}{t^2}\right]\Bigg\}, \tag{11}$$

where $\sigma_3 = n_3 e^2\tau/m$, $n_3 = p_F^3/3\pi^2$, $\Omega = \omega\tau$, $t = qv_F\tau/(1-i\Omega)$, $\mathbf{h} = \mathbf{H}/H$, $\beta = t^{-1}\arctan t$, and $\gamma = (1+t^2)^{-1}$. The known symmetric (Drude) components of the conductivity tensor, $\sigma_{ij}^{(D)}$, in the same notation have the form [12]

$$\sigma_{ij}^{(D)}(\omega,\mathbf{q}) = \sigma_l^{(3)}\hat{q}_i\hat{q}_i + \sigma_{tr}^{(3)}(\delta_{ij}-\hat{q}_i\hat{q}_i), \tag{12}$$

$$\sigma_l^{(3)} = \frac{3\sigma_3}{1-i\Omega}\left(\frac{1-\beta}{t^2}\right)\frac{-i\Omega}{1-\beta-i\Omega}, \quad \sigma_{tr}^{(3)} = \frac{3\sigma_3}{2(1-i\Omega)}\left(\beta-\frac{1-\beta}{t^2}\right).$$

Applied to 2D electron system, our method yields

$$\sigma_{ij}^{(H)}(\omega,\mathbf{q}) = e_{ijn}h_n\sigma_2^H, \quad \sigma_2^H = \sigma_2\frac{\omega_c\tau}{(1-i\Omega)^2}\left(\frac{i\Omega}{1+t^2}\right)\frac{1}{\sqrt{1+t^2}(1-i\Omega)-1}, \tag{13}$$

$$\sigma_{ij}^{(D)}(\omega,\mathbf{q}) = \sigma_l^{(2)}\hat{q}_i\hat{q}_i + \sigma_{tr}^{(2)}(\delta_{ij}-\hat{q}_i\hat{q}_i), \tag{14}$$

$$\sigma_{tr}^{(2)} = \frac{2\sigma_2}{1-i\Omega}\left(\frac{\sqrt{1+t^2}-1}{t^2}\right), \quad \sigma_l^{(2)} = \frac{2\sigma_2}{1-i\Omega}\left(\frac{i\Omega}{t^2}\right)\frac{\sqrt{1+t^2}-1}{1-(1-i\Omega)\sqrt{1+t^2}}$$

($\sigma_2 = n_2 e^2\tau/m$, $n_2 = p_F^2/2\pi$) which agrees with the result recently obtained [6] by means of the 2D classical Boltzmann equation with a modified collision integral.

$$\sigma_{ij}^{(H)}(\omega,\mathbf{q}) = \frac{e^2}{\pi}\left(\frac{ie}{4mc}\right)\times$$

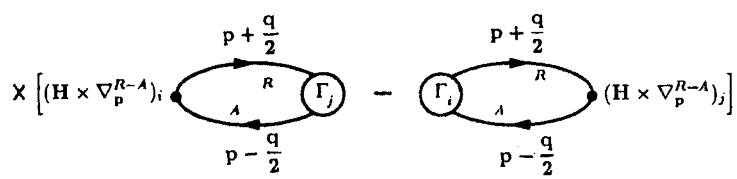

Fig. 1. The Feynman diagrams for the Hall conductivity. The gradient operator $\nabla_{\mathbf{p}}^{R-A}$ in the second term acts to the left on the pair of the Green functions $G^R G^A$. $\mathbf{\Gamma}$ is the usual impurity-renormalized velocity-vertex

In conclusion, we have proposed a method for microscopic calculation of the Hall conductivity in weak magnetic fields. In essence, the method is solely based on the Schwinger's formula, i.e., on the gauge transformation rules in quantum mechanics. The diagrammatic expression for σ_{ij}^H obtained appears to be topologically similar to that for the Drude conductivity. Our approach complements the previous one [2] in that it gives $\sigma(\omega,\mathbf{q})$ at finite $\mathbf{q}$ but only for the p-independent scattering, while the method [2] gives $\sigma(\omega,0)$ but for arbitrary impurity scattering. In addition, the method reported here is hoped to provide a better handle in studying the effects of weak localization and (short-ranged) interparticle interaction on the Hall conductivity.

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