Pis'ma v ZhETF, vol.67, iss.3, pp.166 - 171

THE CONFORMAL ANOMALY ASSOCIATED WITH OPERATOR PRODUCT ACTING IN RANK 1 SYMMETRIC SPACES

A.A.Bytsenko[1)], *A.E.Goncalves*[*1)], *F.L.Williams*[+1)]

St.Petersburg State Technical University, 195251 St.Petersburg, Russia

[*] *Departamento de Fisica, Universidade Estadual de Londrina Caixa Postal 6001, Londrina-Parana, Brazil*

[+] *University of Massachusetts, 01003 Massachusetts, USA*

Submitted 5 January 1998

We compute the conformal anomaly and a contribution to the one-loop effective action associated with the product $\bigotimes_p \mathcal{L}_p$ of the Laplace operators $\mathcal{L}_p$, $p = 1, 2$, acting in irreducible rank 1 symmetric spaces. The explicit form of the zeta functions and the conformal anomaly of the stress-energy momentum tensor is derived.

PACS: 02.40.Vh, 04.60.-m, 04.62.+v

1. It is known that the anomaly associated with multiplicative properties of regularized determinants of (pseudo-) differential operators can be expressed by means of the non-commutative residue, the Wodzicki residue [1] (see also Refs. [2, 3]). The Wodzicki residue, which is the unique extension of the Dixmier trace to the wider class of (pseudo-) differential operators [4, 5], has been considered within the non-commutative geometrical approach to the standard model of the electroweak interactions [6 – 11] and the Yang – Mills action functional. Some recent papers along these lines can be found in Refs. [12 – 14]. The product of two (or more) differential operators of Laplace type can arise in higher derivative field theories (for example, in higher derivative quantum gravity [15, 16]). The partition function corresponding to the product of two elliptic second order differential operators for the simplest $O(2)$ invariant model of self-interacting charged fields in $\mathbb{R}^4$ [17] has been derived recently in Ref. [18].

Note also that the conformal deformations of a metric and the corresponding conformal anomaly can play an important role in quantum theories with higher derivatives. In fact evaluation of the conformal anomaly is actually possible only for even dimensional spaces and up to now its computation is extremely involved. The general structure of such anomaly in curved d-dimensional spaces (d even) has been studied in Ref. [19]. We briefly mention here analysis related to this phenomenon for constant curvature spaces. The conformal anomaly calculation for d-dimensional sphere can be found in Ref. [20, 21]. The explicit computation of the anomaly (of the stress-energy tensor) for scalar and spinor quantum fields in d-dimensional compact hyperbolic spaces has been carried out in Ref. [22] (see also Refs. [23, 24]) using zeta-function regularization and the Selberg trace formula techniques.

2. The purpose of this letter is to analyze a contribution to the effective action (in general form) and the conformal anomaly associated with the product $(\bigotimes_p \mathcal{L}_p)$, where $\mathcal{L}_p$, $p = 1, 2$, are the Laplace operators acting in general rank 1 symmentric spaces. We

1) e-mail: abyts@spin.hop.stu.neva.ru, goncalve@fisica.uel.br, williams@math.umass.edu

shall be working with irreducible rank 1 symmetric spaces $M \equiv X = G/K$ of non-compact type. Thus G will be a connected non-compact simple split rank 1 Lie group with finite center and $K \subset G$ a maximal compact subgroup. The following representations of X up to local isomorphism can be chosen

$$X = \begin{bmatrix} SO_1(n,1)/SO(n) & \text{(I)} \\ SU(n,1)/U(n) & \text{(II)} \\ SP(n,1)/(SP(n)\otimes SP(1)) & \text{(III)} \\ F_{4(-20)}/Spin(9) & \text{(IV)} \end{bmatrix}, \tag{1}$$

where $n \geq 2$, and $F_{4(-20)}$ is the unique real form of F_4 (with Dynkin diagram $\circ - \circ = \circ - \circ$) for which the character ($\dim X - \dim K$) assumes the value (-20) [25]. We assume that if G_1 or $G_2 = SO(m,1)$ or $SU(q,1)$ then m is even and q is odd.

We shall need further a suitable regularization of the determinant of an elliptic differential operator, and shall make the choice of zeta-function regularization. The spectral zeta function associated with the product $\bigotimes \mathcal{L}_p$ has the form

$$\zeta(s|\bigotimes_p \mathcal{L}_p) = \sum_{j\geq 0} n_j \prod_p^2 (\lambda_j + b_p)^{-s}, \tag{2}$$

where $\{\lambda_l\}_{l=0}^{\infty}$ is the set of eigenvalues of the Laplace operator L, while $n_l(\chi)$ denote the multiplicity of λ_l; $\mathcal{L}_p \equiv L + b_p$, and b_p are arbitrary constants. We shall always assume that $b_1 \neq b_2$, say $b_1 > b_2$. If $b_1 = b_2$ then $\zeta(s|\bigotimes \mathcal{L}_p) = \zeta(2s|\mathcal{L})$ is a well-known function.

For $Res > \frac{d}{4}$ the explicit meromorphic continuation holds [26]:

$$\zeta(s|\bigotimes_p \mathcal{L}_p) = A \sum_{j=0}^{d/2-1} a_{2j}\left(\mathcal{F}_j(s) - E_j(s)\right) + \mathcal{I}(s), \tag{3}$$

where

$$E_j(s) = 4\int_0^{\infty} \frac{dr r^{2j+1}}{1+e^{2a(G)r}} \prod_p (r^2 + B_p)^{-s}, \tag{4}$$

which is an entire function of s and

$$\mathcal{F}_j(s) = (B_1B_2)^{-s} \frac{j!\left(\frac{2B_1B_2}{B_1+B_2}\right)^{j+1} F\left(\frac{j+1}{2}, \frac{j+2}{2}; s+\frac{1}{2}; \left(\frac{B_1-B_2}{B_1+B_2}\right)^2\right)}{(2s-1)(2s-2)...(2s-(j+1))}, \tag{5}$$

$$\mathcal{I}(s) = (b_1 - b_2)^{1/2-s}\frac{\sqrt{\pi}}{\Gamma(s)}\int_0^{\infty} dt\theta(t) I_{s-1/2}(\frac{b_1-b_2}{2}t)t^{s-1/2}. \tag{6}$$

In Eqs. (3) - (6) $B_p = \rho_0^2 + b_p$, $\rho_0 = (n-1)/2, n, 2n+1, 11$ in the case (I)–(IV) respectively in Eq. (1), $F(\alpha,\beta;\gamma;z)$ is the hypergeometric function, $I_\nu(z)$ are the Bessel functions, $\theta(t)$ is an entire function of t, and A is some constant (for more detail see Ref. [26]). The suitable Harish – Chandra – Plancherel measure is given as follows:

$$|C(r)|^{-2} = C_G \pi r P(r) \tanh\left(a(G)r\right) = C_G\pi \sum_{l=0}^{d/2-1} a_{2l} r^{2l+1} \tanh\left(a(G)r\right), \tag{7}$$

where

$$a(G) = \begin{bmatrix} \pi & \text{for } G = SO_1(2n,1) & \\ \frac{\pi}{2} & \text{for } G = SU(q,1), & q \text{ odd} \\ & \text{or } G = SP(m,1), & F_{4(-20)} \end{bmatrix}, \tag{8}$$

while C_G is some constant depending on G, and where the $P(r)$ are even polynomials (with suitable coefficients a_{2l}) of degree $d-2$ for $G \neq SO(2n+1,1)$, and of degree $d-1 = 2n$ for $G = SO_1(2n+1,1)$ [24, 27].

The goal now is to compute the zeta function $\zeta(s|\bigotimes_p \mathcal{L}_p)$ and its derivative at $s=0$. Thus we have

$$\zeta(0|\bigotimes_p \mathcal{L}_p) = A \sum_{j=0}^{d/2-1} \frac{(-1)^{j+1}}{2(j+1)} a_{2j} \left\{ \sum_l^2 B_l^{j+1} + (2-2^{-2j}) \left[\frac{\pi}{a(G)}\right]^{2j+2} \mathcal{B}_{2j+2} \right\}, \tag{9}$$

$$\zeta'(0|\bigotimes_p \mathcal{L}_p) = A \sum_{j=0}^{d/2-1} a_{2j} \sum_l^4 \mathcal{E}_l, \tag{10}$$

where

$$\mathcal{E}_1 = j!(B_1^{j+1} + B_2^{j+1}) \sum_{k=0}^{j} \frac{(-1)^{k+1}}{k!(j-k)!(j+1-k)!}, \tag{11}$$

$$\mathcal{E}_2 = B_2^{j+1} \left(\frac{B_1 - B_2}{2B_1}\right) \frac{(-1)^j}{(j+1)!} \sum_{k=1}^{\infty} \frac{(j+k+1)!}{(k+1)!} \sigma_k \left(\frac{B_1 - B_2}{B_1}\right)^k, \tag{12}$$

$$\mathcal{E}_3 = \log(B_1 B_2) \frac{(-1)^j}{2(j+1)} (B_1^{j+1} + B_2^{j+1}) - 4 \int_0^{\infty} \frac{dr r^{2j+1} \log\left(\frac{r^2+B_1}{r^2+B_2}\right)}{1 + e^{2a(G)r}}, \tag{13}$$

$$\mathcal{E}_4 \equiv \mathcal{I}'(s=0), \tag{14}$$

and $\sigma_k = \sum_{k=1}^{n} k^{-1}$.

3. After a standard functional integration in scalar theory the contribution to the Euclidean one-loop effective action can be written as follows

$$W^{(1)} = \frac{1}{2} \log\det(\bigotimes_p \mathcal{L}_p/\mu^2) = -\frac{1}{2}[\zeta'(0|\bigotimes_p \mathcal{L}_p) + \log\mu^2 \zeta(0|\bigotimes_p \mathcal{L}_p)], \tag{15}$$

where μ^2 is a normalization parameter. As a result we have

$$W^{(1)} = -\frac{1}{2} A \sum_{j=0}^{d/2-1} a_{2j} \left[\sum_l^4 \mathcal{E}_l + \log\mu^2 (\mathcal{F}_j(0) - E_j(0)) \right], \tag{16}$$

where

$$\mathcal{F}_j(0) = \frac{(-1)^{j+1}}{2(j+1)} \sum_l^2 B_l^{j+1}, \tag{17}$$

$$E_j(0) = \frac{(-1)^j}{j+1} (1 - 2^{-2j-1}) \left[\frac{\pi}{a(G)}\right]^{2j+2} \mathcal{B}_{2j+2}, \tag{18}$$

$\mathcal{I}(0) = 0$, and $\mathcal{B}_{2n}$ are the Bernoulli numbers.

4. In this section we start with a conformal deformation of a (pseudo-) Riemannian metric and the conformal anomaly of the energy stress tensor. For constant conformal deformations the variation of the connected vacuum functional (effective action) can be expressed in terms of the generalized zeta function related to an elliptic self-adjoint operator $\mathcal{O}$ [15]

$$\delta W = -\zeta(0|\mathcal{O}) \log \mu^2 = \int_M dx < T_{\mu\nu}(x) > \delta g^{\mu\nu}(x), \tag{19}$$

where $< T_{\mu\nu}(x) >$ means that all connected vacuum graphs of the stress-energy tensor $T_{\mu\nu}(x)$ are to be included. Therefore the Eq. (19) leads to

$$< T_\mu^\mu(x) >= (\text{Vol}M)^{-1}\zeta(0|\mathcal{O}). \tag{20}$$

In the case of sphere S^d of unit radius we have for example $(\text{Vol}S^d) = 2\pi^{(d+1)/2}/\Gamma((d+ +1)/2)$, while the Eq. (3) gives $AC_G = (\text{Vol}M)[(4\pi)^{d/2}\Gamma(d/2)]^{-1}$ (see for detail Ref. [24]). As a result we have $(\text{Vol}M) = A(4\pi)^{d/2}\Gamma(d/2)$.

The formulae (3), (17) and (18) give an explicit result for the conformal anomaly, namely

$$< T_\mu^\mu(x) >_{(\mathcal{O}=\bigotimes \mathcal{L}_p)} = \frac{1}{(4\pi)^{d/2}\Gamma(d/2)} \sum_{j=0}^{d/2-1} \frac{(-1)^{j+1}}{2(j+1)} a_{2j} \times$$
$$\times \left\{ \sum_l^2 B_l^{j+1} + (2-2^{-2j}) \left[\frac{\pi}{a(G)}\right]^{2j+2} \mathcal{B}_{2j+2} \right\}, \tag{21}$$

where d is even.

For $B_1 = B_2 = B$ the anomaly (21) is associated with Laplace operator $\mathcal{L} = L + b$ and has the form

$$< T_\mu^\mu(x) >= \frac{1}{(4\pi)^{d/2}\Gamma(d/2)} \sum_{j=0}^{\frac{d}{2}-1} \frac{(-1)^{j+1}}{2(j+1)} a_{2j} \left\{ B^{j+1} + (2-2^{-2j}) \left[\frac{\pi}{a(G)}\right]^{2j+2} \mathcal{B}_{2j+2} \right\}. \tag{22}$$

Note that for minimally coupled scalar field of mass m, $B = \rho_0^2 + m^2$.

The simplest case is, for example $G = SO_1(2,1) \simeq SL(2,\mathbf{R})$; besides $X = H^2$ is a two-dimensional real hyperbolic space. Then we have $\rho_0^2 = 1/4, a_{20} = 1, C_G = 1, a(G) = = \pi, |C(r)|^{-2} = \pi r \tanh(\pi r)$, and finally $< T_\mu^\nu(x \in H^2) >= -(b + 1/3)/(4\pi)$.

For real d-dimensional hyperbolic space $C_G = [2^{d-2}\Gamma(d/2)]^{-2}$, while the scalar curvature is $R(x) = -d(d-1)$. For the conformally invariant scalar field we have $B = \rho_o^2 + R(x)(d-2)/[4(d-1)]$. As a consequence, for all constant curvature spaces $B_1 = B_2$; for hyperbolic spaces $B = 1/4$ and

$$< T_\mu^\mu(x \in H^d) >= \frac{1}{(4\pi)^{d/2}\Gamma(d/2)} \sum_{j=0}^{d/2-1} \frac{(-1)^{j+1}}{j+1} a_{2j} \left\{2^{-2j-2} + (1-2^{-2j-1})\mathcal{B}_{2j+2}\right\}. \tag{23}$$

Thus in conformally invariant scalar theory the anomaly of the stress tensor coincides with one associated with operator product. This statement holds not only for hyperbolic spaces considered above but for all constant curvature manifolds as well.

5. In this paper the one-loop contribution to the effective action (16) and the conformal anomaly of the stress-energy momentum tensor (21) related to the operator product have been evaluated explicitly. In addition we have considered the product $\bigotimes_p \mathcal{L}_p$ of Laplace operators $\mathcal{L}_p$ acting in irreducible rank 1 symmetric spaces. As an example the conformal anomaly has been computed for real d-dimensional hyperbolic spaces.

We have shown that for the class of constant curvature manifolds the conformal anomalies associated with the Laplace type operator $\mathcal{L}$ and the product $\bigotimes_p \mathcal{L}_p$ coincide. Our formulae can be generalized to the case of transverse and traceless tensor fields and spinors in real hyperbolic spaces (see, for example, [28 – 30]). Indeed in the case of the spin-1 field (vector field theory) for instance we should draw attention to the fact that the Hodge – de Rham operator $-(d\delta + \delta d)$ acting on co-exact one-forms corresponds to the mass operator $(L + d - 1)g_{\mu\nu}(x)$. The eigenvalues of this operator are $\lambda_l + (\rho_0 - 1)^2$, and for the Proca field of mass m we have $B = (\rho_0 - 1)^2 + m^2$. Finally we have also computed the anomaly in a simple situation, namely for the conformally invariant scalar fields. Our result (23) coincides with the quantum correction reported in Ref. [22] for compact hyperbolic spaces. Recently the conformal anomaly of dilaton coupled matter in four dimensions has been calculated in Refs. [31, 32]. It would be of great interest to generalize our results to the dilaton dependent trace anomaly.

An extension of the above evaluation of the effective action and the conformal anomaly for higher spin fields seems to us certainly feasible. The analysis of multiplicative properties of Laplace type operators and related zeta functions will be interesting in view of future applications to concrete problems in quantum field theory and for mathematical applications as well.

A.A. Bytsenko wishes to thank CNPq and the Department of Physics of Londrina University for financial support and kind hospitality. The research of A.A. Bytsenko was supported in part by Russian Foundation for Fundamental Research Grant 98-02-18380-a and by Russian Universities Grant 6-18-1997.

---

1. M.Wodzicki, *Non-Commutative Residue*, Ch. In Lecture Notes in Mathematics, Eds. Yu.I.Manin, Springer-Verlag, Berlin **1289**, 320 (1987).
2. M.Kontsevich and S.Vishik, Preprint MPI/94-30 (1994).
3. M.Kontsevich and S.Vishik, hep-th/9406140 (1994).
4. A.Connes, Commun. Math. Phys. **117**, 673 (1988).
5. D.Kastler, Commun. Math. Phys. **166**, 633 (1995).
6. A.Connes and J. Lott, Nucl. Phys. **B 18**, 29 (1990).
7. A.Connes, *Non-Commutative Geometry*, Academic Press, New York, 1994.
8. A.Connes, hep-th/9603053 (1996).
9. A.H.Chamseddine and A. Connes, hep-th/9606001 (1996).
10. A.H.Chamseddin and A. Connes, hep-th/9606056 (1996).
11. C.P.Martin, J.M.Gracia-Bondia and J.C. Varilly, hep-th/9605001 (1996).
12. W.Kalau and M.Walze, J. Geom. Phys. **16**, 327 (1995).
13. T.Ackermann, funct-an/9506006 (1995).
14. T.Ackermann and J. Tolksdorf, hep-th/9503152 (1995).
15. N.D.Birrell and P.C.W.Davies, *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge, 1982.
16. I.L.Buchbinder, S.D.Odintsov, and I.L.Shapiro, *Effective Action in Quantum Gravity*, A. Hilger, Bristol, 1992.
17. K.Benson, J.Bernstein, and S.Dodelson, Phys. Rev. **D 44**, 2480 (1991).
18. E.Elizalde, L.Vanzo, and S. Zerbini, hep-th/9701060 (1997).
19. S.Deser and A.Schwimmer, Phys. Lett. **B309**, 279 (1993).
20. E.Copeland and D.Toms, Class. Quant. Grav. **3**, 431 (1986).
21. V.P.Dergalev and S.D.Odintsov, *Effective Action of Conformally-Invariant Theories in Multidimensional Space*, In Gravity and Fundamental Interactions, Moscow 1988.

22. A.A.Bytsenko, E.Elizalde and S.D.Odintsov, JMP **36**, 5084 (1995).
23. E.Elizalde, S.D.Odintsov, A.Romeo et al., *Zeta Regularization Techniques with Applications*, World Scientific, Singapore, 1994.
24. A.A.Bytsenko, G.Cognola, L.Vanzo and S.Zerbini, Phys. Rep. **266**, 1 (1996).
25. S.Helgason, *Differential Geometry and Symmetric Spaces*, Pure and Applied Math. Ser. **12**, Academic Press, 1962.
26. A.A.Bytsenko and F.L.Williams, hep-th/9706120 (1997).
27. F.L.Williams, JMP **38**, 796 (1997).
28. R.Camporesi, Commun. Math. Phys. **148**, 283 (1992).
29. R.Camporesi and A.Higuchi, Phys. Rev. **D 47**, 3339 (1993).
30. R.Camporesi and A.Higuchi, JMP **35**, 4217 (1994).
31. S.Nojiri and S.D.Odintsov, hep-th/9706009 (1997).
32. S.Nojiri and S.D.Odintsov, hep-th/9706143 (1997).