Supplemental Material to the article

“Relaxation of Coherent Excited States of a Superconductor to a Superconducting Reservoir”

The system of equations (3)–(8) of the main part of the article, which describes the interaction of a diffuse superconducting strip with microwave radiation and a superconducting reservoir, is a nontrivial system of equations, which consists of equations in finite differences (Eqs. (3)–(6) of the main part of the article), an integral equation (Eq. (7) of the main part of the article), and an algebraic equation (Eq. (8) of the main part of the article). However, due to conditions Eq. (1) of the main part of the article, normalized radiation intensity α can be used as a small parameter, and we can use the linear expansion for the Green's functions and the order parameter, for example: $\breve{G}_0 = \breve{G}_0^0 + \delta\breve{G}_0 = \breve{G}_0^0 + \partial_\alpha \breve{G}_0 \alpha$. Linearization of Eqs. (3)–(8) of the main part of the article makes it possible to simply find their solution, as we show it further.

The first iteration on a small parameter α of the solution of the kinetic equation, together with the collision integrals (Eqs. (4)–(6) of the main part of the article), immediately leads to linear by α solution for the change of distributon function δf_L, presented by Eq. (10) of the main part of the article.

Linearization of the equation for the spectral functions (Eq. (3) of the main part of the article) leads to the following relation:

$$-i(E + i\Gamma G_{r,0}^R)\delta F_0^R - i(\Delta_0^0 - i\Gamma F_{r,0}^R)\delta G_0^R - iG_0^{R,0}\delta\Delta_0 + \alpha\Pi_0 = 0, \tag{1}$$

with $\Pi_0 = \left\{ \left(G_{0+}^{R,0} + G_{0-}^{R,0} \right) F_0^{R,0} + \left(F_{0+}^{R,0} + F_{0-}^{R,0} \right) G_0^{R,0} \right\}$.

Linearization of the normalization condition (Eq. (8) of the main part of the article) allows us to link variations of functions F_0^R and G_0^R:

$$\delta G_0^R = \frac{F_0^{R,0}}{G_0^{R,0}} \delta F_0^R. \tag{2}$$

Substitution of Eq. (2) into Eq. (1) allows us to represent its solution in the form of the full derivative:

$$\delta F_0^R = \frac{\partial F_0^R}{\partial \Delta_0}_{|\alpha=0} \delta\Delta_0 + \frac{\partial F_0^R}{\partial \alpha}_{|\Delta=\Delta_0^0} \alpha. \tag{3}$$

The partial derivatives in Eq. (3) are given by the following expressions:

$$\frac{\partial F_0^R}{\partial \alpha}_{|\Delta=\Delta_0^0} = \frac{-i\Pi_0 \left(E + i\Gamma G_{r,0}^R \right)}{\left\{ \left(E + i\Gamma G_{r,0}^R \right)^2 - \left(\Delta_0^0 - i\Gamma F_{r,0}^R \right)^2 \right\}}, \tag{4}$$

and

$$\frac{\partial F_0^R}{\partial \Delta_0}_{|\alpha=0} = \frac{-G_0^{R,0} \left(E + i\Gamma G_{r,0}^R \right)}{\left\{ \left(E + i\Gamma G_{r,0}^R \right)^2 - \left(\Delta_0^0 - i\Gamma F_{r,0}^R \right)^2 \right\}}. \tag{5}$$

Equation (3) expresses the linear change of the anomalous Green's functions δF_0^R of a superconducting strip under the influence of the microwave radiation, taking into account relaxation to superconducting reservoir, as a sum of two terms: the first, proportional to the variation of the order parameter $\delta\Delta_0$, and the second, proportional to the normalized signal intensity α.

The relation for the small correction of the order parameter $\delta\Delta_0$ of a superconducting strip under the influence of the microwave radiation, taking into account relaxation to superconducting reservoir, one can find after linearization of the self-consistency equation (Eq. (7) of the main part of the article):

$$\delta\Delta_0 = \lambda_{ep} \frac{\{\delta_F \Delta_0 + \delta_{f_L} \Delta_0\}}{\left\{ 1 + \lambda_{ep} \int_0^{\hbar\omega_D} dE \frac{\partial \mathrm{Re}\left[F_0^R\right]}{\partial \Delta_0}_{|\alpha=0} f_L^0 \right\}}, \tag{6}$$

with

$$\delta_F \Delta_0 = -\alpha \int_0^{\hbar\omega_D} dE \frac{\partial \mathrm{Re}\left[F_0^R\right]}{\partial \alpha}_{|\Delta=\Delta_0^0} f_L^0, \tag{7}$$

and

$$\delta_{f_L} \Delta_0 = -\alpha \int_0^{\hbar\omega_D} dE \mathrm{Re}\Big[F_0^{R,0}\Big] \frac{\partial f_L}{\partial \alpha}_{|\Delta=\Delta_0^0}. \tag{8}$$

The first term in the numerator of Eq. (6) $\delta_F \Delta_0$ (Eq. (7)) describes the change of the order parameter Δ_0 due to the change of the spectral function F_0^R. The second term in the numerator of Eq. (6) $\delta_{f_L} \Delta_0$ (Eq. (8)) describes the change of the order parameter Δ_0 due to the change of the distribution function f_L. In equation (6) we omit the term with partial derivative $\frac{\partial f_L}{\partial \Delta_0}_{|\alpha=0}$ because this derivative identically equals to zero. Substituting in place of the one in the denominator of Eq. (6) the relation: $1 = -\lambda_{ep} \int\limits_0^{\hbar\omega_D} d\left(\frac{\varepsilon}{\Delta_0}\right) f_L(\varepsilon) \mathrm{Re}\left[F_0^R(\varepsilon)\right]$, which follows from the self-consistency equation (Eq. (7) of the main part of the article), one can get the expression for the variation $\delta\Delta_0$ which does not contain the constant of electron-phonon interaction λ_{ep}:

$$\delta\Delta_0 = \frac{(\delta_F \Delta_0 + \delta_{f_L} \Delta_0)}{J_1}, \tag{9}$$

with

$$J_1 = \int_0^{\hbar\omega_D} dE \mathrm{Re}\Big[\frac{\partial F_0^R}{\partial \Delta_0}_{|\alpha=0} - \frac{F_0^R}{\Delta_0}\Big] f_L^0. \tag{10}$$

The integrand in Eq. (10) converges rapidly ($O(1/E^2)$), which allows us to extend the upper integral limit up to infinity. In the considered limit of low temperatures $k_B T \ll \hbar\omega_0$, $f_L^0 = 1$ at $E > 0$ in Eq. (10), and the expression for J_1 Eq. (10) coincides with that used in the main text of the article. In the previously considered case of relaxation to a normal reservoir [12], the analytical calculation of J_1 gave us a value very close to 1. In the current case of relaxation to a superconducting reservoir, the J_1 value should be calculated numerically using Eq. (10).

In the same way the change of the order parameter Δ_0 due to the change of the spectral functions $\delta_F \Delta_0$ must be calculated now numerically using Eqs. (4), (7) with $f_L^0 = 1$, which leads to the following expression: $\delta_F \Delta_0 = -\alpha J_0$, where J_0 is defined in the main text of the article. The analytical result $\delta_F \Delta_0 = -\pi\alpha$ of the paper [12] is correct in the case of relaxation to a reservoir from a normal metal only and can not be used now.

To calculate the change of the order parameter Δ_0 due to the change of the distribution function $\delta_{f_L} \Delta_0$, knowledge of the derivative $\frac{\partial f_L}{\partial \alpha}_{|\Delta=\Delta_0^0}$ is necessary, as followed from Eq. (8). Easy to see that this derivative $\frac{\partial f_L}{\partial \alpha}_{|\Delta=\Delta_0^0}$ is given by the right hand side of the Eq. (10) of the main part of the article. Numerical calculations based on the Eq. (8) and the Eq. (10) of the main part of the article give values of $\delta_{f_L} \Delta_0$ much less than values of $\delta_F \Delta_0$ for any reasonable set of parameters of the problem. The explanation for this fact is the same as in the case of relaxation to a normal metal reservoir [12]. In the low-temperature limit $k_B T \ll \hbar\omega_0$ the differences $f_{L\pm}^0 - f_L^0$, which enter in the relation for $\frac{\partial f_L}{\partial \alpha}_{|\Delta=\Delta_0^0}$ (Eq. (10) of the main part of the article) are unequal to zero only in the small energy interval $-\hbar\omega_0 < E < \hbar\omega_0$. The integrand in the expression for $\delta_F \Delta_0$ Eq. (7) has no such restrictions. Therefore, by virtue of the second inequality in Eq. (1) of the main part of the article $\delta_{f_L} \Delta_0 \ll \delta_F \Delta_0$ and $\delta_{f_L} \Delta_0$ can be neglected compare to the $\delta_F \Delta_0$ in Eq. (9).

Finally, the expression for $\delta\Delta_0$ becomes the following:

$$\delta\Delta_0 = \kappa\alpha, \tag{11}$$

with $\kappa - \frac{J_0}{J_1}$.

Substitution of Eqs. (2)–(5), (11) into the expression for corrections to the density of states of a superconducting strip $\delta N = \mathrm{Re}\big[\delta G_0^R\big]$ allows us to obtain Eq. (9) of the main text of the article.