Supplemental Material to the article
“Non-reciprocal propagation of solitons in a chiral medium”

Application of the technique of the inverse scattering transform to the system (7), (10), (11) in the main text follows to well known approach, see, for instance, in [1, 2]. The solutions of the spectral problem (14) in the main text have the involution

$$\Phi = \mathbf{M}\Phi(\lambda^*)^*\mathbf{M}^{-1}, \tag{1}$$

where

$$\mathbf{M} = \begin{pmatrix} 0 & -\varepsilon\frac{\lambda+f}{\lambda-f} \\ 1 & 0 \end{pmatrix}. \tag{2}$$

The respective Jost functions $\Phi_{1,2}$ corresponded to decaying as $\tau \to \pm\infty$ potential and its derivatives for ground state $F_3(\varsigma) = 1, F(\varsigma) = 0$ posses the following asymptotics:

$$\Phi = \exp\left(-i\lambda\sigma_3\varsigma\right), \quad \tau \to \pm\infty. \tag{3}$$

The symmetry properties (1) correspond to the following respective matrix forms of the Jost functions:

$$\Phi = \begin{pmatrix} \psi_1^\pm & -\varepsilon\psi_2^{\pm\,*}\frac{\lambda+f}{\lambda-f} \\ \psi_2^\pm & \psi_1^{\pm\,*} \end{pmatrix}. \tag{4}$$

Respective functions are related by the scattering matrices $\hat{S}$

$$\Phi^- = \Phi^+\hat{S}, \tag{5}$$

where

$$\hat{S} = \begin{pmatrix} a & b^* \\ -\varepsilon b\left(\lambda - f\right)/\left(\lambda + f\right) & a^* \end{pmatrix}. \tag{6}$$

The Jost functions have the presentation

$$\Phi^+(\theta) = \mathrm{e}^{-i\lambda\sigma_3\theta} + \int_\theta^\infty \begin{pmatrix} \lambda K(\varsigma, s) & \varepsilon\left(\lambda + f\right)Q_2^*(\varsigma, s) \\ \left(\lambda - f\right)Q_2(\varsigma, s) & \lambda K_2^*(\varsigma, s) \end{pmatrix} \mathrm{e}^{-i\lambda\sigma_3 s}\, d\,s. \tag{7}$$

Using these presentations of the Jost functions we derive from the spectral problem (14) in the main text and (3)

$$F(\varsigma) = \frac{2\left[1 - iK(\varsigma, \varsigma)\right]Q^*(\varsigma, \varsigma)}{\left[1 - iK(\varsigma, \varsigma)\right]\left[1 + iK^*(\varsigma, \varsigma)\right] + \varepsilon|Q(\varsigma, \varsigma)|^2}, \tag{8}$$

$$F_3(\varsigma) = \frac{\left[1 - iK(\varsigma, \varsigma)\right]\left[1 + iK^*(\varsigma, \varsigma)\right] - \varepsilon|Q(\varsigma, \varsigma)|^2}{\left[1 - iK(\varsigma, \varsigma)\right]\left[1 + iK^*(\varsigma, \varsigma)\right] + \varepsilon|Q(\varsigma, \varsigma)|^2}. \tag{9}$$

The Marchenko-type equations are:

$$\varepsilon\left(f - i\partial_y\right)Q(\varsigma, y) + \mathcal{G}(\varsigma + y) = \int_\varsigma^\infty K^*(\varsigma, s)i\partial_y\mathcal{G}(s + y)ds, \tag{10}$$

$$i\partial_y K(\varsigma, y) = -\int_\varsigma^\infty Q^*(\varsigma, s)\left(f + i\partial_y\right)\mathcal{G}(s + y)ds, \tag{11}$$

where

$$\mathcal{G}(y) = \int_{\mathcal{C}} \frac{b(\chi)\mathrm{e}^{i\lambda y}}{a}\frac{d\lambda}{2\pi i}, \tag{12}$$

$\mathcal{C}$ is the contour of integration in the upper half-place.

The scattering data dependence vs χ for $S_3(\chi, \varsigma) = S_0, S(\chi, \varsigma) = 0, \varsigma \to \pm\infty$, is determined by function

$$b(\chi) = b(0) \exp\left\{ iS_0 \left[\frac{(bq + 4\eta)\left(\left(b^2 - 2\right) q + 4b\eta\right)}{2(bq + 4\eta + 2)} \right] \right\}, \tag{13}$$

here η is a complex number in the upper half-plane.

References

[1] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of Solitons: The Inverse Scattering Method*, Springer-Verlag (1984).

[2] A. A. Zabolotskii, Phys. Rev. A **85**, 063833 (2012).