Supplemental Material to the article "Two-dimensional Coulomb glass as a model for vortex pinning in superconducting films"

1. Schwinger–Dyson identities. The Equation (2) from the main paper text is a good starting point for the diagram technique in terms of auxiliary field φ. It would then be useful to derive exact identities which relate its correlation functions to the correlation functions of vortex density.

The arbitrary correlation function is defined as follows:

$$\langle O[\delta n, \varphi]\rangle \equiv \int \mathcal{D}\varphi \mathrm{Tr}_{\mathrm{v}} O[\delta n, \varphi] e^{-S[\varphi, \delta n]}. \tag{1}$$

Due to the invariance of the integration measure w.r.t. infinitesimal transformations $\varphi^a_{\mathbf{r}} \mapsto \varphi^a_{\mathbf{r}} + \epsilon^a_{\mathbf{r}}$, we immediately obtain:

$$\langle O[\delta n, \varphi]\rangle \equiv \int \mathcal{D}\varphi \mathrm{Tr}_{\mathrm{v}} \left(O[\delta n, \varphi] + \sum_{\mathbf{r}} \epsilon^a_{\mathbf{r}} \left[\frac{\partial O[\delta n, \varphi]}{\partial \varphi^a_{\mathbf{r}}} - O[\delta n, \varphi] \frac{\partial S[\varphi, \delta n]}{\partial \varphi^a_{\mathbf{r}}} \right] \right) e^{-S[\varphi, \delta n]}, \tag{2}$$

and due to arbitrary value of $\epsilon_{\mathbf{r}}$, taking also into account the exact form of the action Eq. (2), we immediately obtain the following identity:

$$\left\langle \frac{\partial O[\delta n, \varphi]}{\partial \varphi^a_{\mathbf{r}}} \right\rangle = \left\langle O[\delta n, \varphi] \frac{\partial S[\varphi, \delta n]}{\partial \varphi^a_{\mathbf{r}}} \right\rangle = \left\langle O[\delta n, \varphi] \left\{ \sum_{\mathbf{r}_1} (\beta \hat{J})^{-1}_{\mathbf{r}\mathbf{r}_1} \varphi^a_{\mathbf{r}_1} - i \delta n^a_{\mathbf{r}} \right\} \right\rangle. \tag{3}$$

By picking out various O, we can obtain various useful identities for the correlation functions. In particular, we have:

$$O[\delta n, \varphi] = \varphi^b_{\mathbf{r}'} \Rightarrow \delta_{ab}\delta_{\mathbf{r}\mathbf{r}'} = \sum_{\mathbf{r}_1} (\beta \hat{J})^{-1}_{\mathbf{r}\mathbf{r}_1} \left\langle \varphi^a_{\mathbf{r}_1} \varphi^b_{\mathbf{r}'} \right\rangle - i \left\langle \delta n^a_{\mathbf{r}} \varphi^b_{\mathbf{r}'} \right\rangle, \tag{4}$$

$$O[\delta n, \varphi] = i\delta n^b_{\mathbf{r}'} \Rightarrow 0 = \sum_{\mathbf{r}_1} (\beta \hat{J})^{-1}_{\mathbf{r}\mathbf{r}_1} i \left\langle \varphi^a_{\mathbf{r}_1} \delta n^b_{\mathbf{r}'} \right\rangle + \left\langle \delta n^a_{\mathbf{r}} \delta n^b_{\mathbf{r}'} \right\rangle \tag{5}$$

thus the following identity follows:

$$\left\langle \delta n^a_{\mathbf{r}} \delta n^b_{\mathbf{r}'} \right\rangle = \delta_{ab} (\beta \hat{J})^{-1}_{\mathbf{r}\mathbf{r}'} - \sum_{\mathbf{r}_{1,2}} (\beta \hat{J})^{-1}_{\mathbf{r}\mathbf{r}_1} \left\langle \varphi^a_{\mathbf{r}_1} \varphi^b_{\mathbf{r}_2} \right\rangle (\beta \hat{J})^{-1}_{\mathbf{r}_2\mathbf{r}'}. \tag{6}$$

Furthermore one obtains, for the correlation function $\langle \varphi\varphi \rangle$ in the form of Eq. (9):

$$\langle \delta n \delta n \rangle = \frac{\hat{\mathcal{Q}}}{1 + \beta \hat{J} \hat{\mathcal{Q}}}. \tag{7}$$

It is also worth noting that the local correlation function has then the form $\left\langle \delta n^a_{\mathbf{r}} \delta n^b_{\mathbf{r}} \right\rangle = \hat{\mathcal{Q}} - \hat{\mathcal{Q}} \hat{\mathcal{G}} \hat{\mathcal{Q}}$, which, strictly speaking, does not coincide with $\hat{\mathcal{Q}}$; the difference is however parametrically small by an extra $1/W$ factor.

1.1. Polarizability fluctuations. The same procedure allows one to obtain the following expression for the four-point correlation function:

$$\left\langle\left\langle \delta n^a_{\mathbf{r}_1} \delta n^b_{\mathbf{r}_2} \delta n^c_{\mathbf{r}_3} \delta n^d_{\mathbf{r}_4} \right\rangle\right\rangle = (\beta \hat{J})^{-1}_{\mathbf{r}_1\mathbf{r}'_1} (\beta \hat{J})^{-1}_{\mathbf{r}_2\mathbf{r}'_2} (\beta \hat{J})^{-1}_{\mathbf{r}_3\mathbf{r}'_3} (\beta \hat{J})^{-1}_{\mathbf{r}_4\mathbf{r}'_4} \left\langle\left\langle \varphi^a_{\mathbf{r}'_1} \varphi^b_{\mathbf{r}'_2} \varphi^c_{\mathbf{r}'_3} \varphi^d_{\mathbf{r}'_4} \right\rangle\right\rangle. \tag{8}$$

In the vicinity of T_c, the mean-field theory predicts the following form of the correlation function $\left\langle\left\langle \mathcal{G}^{ab}_{\mathbf{r}} \mathcal{G}^{cd}_{\mathbf{r}'} \right\rangle\right\rangle \simeq \left\langle\left\langle \varphi^a_{\mathbf{r}} \varphi^b_{\mathbf{r}} \varphi^c_{\mathbf{r}'} \varphi^d_{\mathbf{r}'} \right\rangle\right\rangle$ (for $|\mathbf{r} - \mathbf{r}'| \gg l$, by definition of $\hat{\mathcal{G}}$ matrix). Using the diagram technique, one can show that coordinate dependence of the φ correlation function can be restored in the limit $|\mathbf{r}'_1 - \mathbf{r}'_2| \lesssim l$ and $|\mathbf{r}'_3 - \mathbf{r}'_4| \lesssim l$ as follows:

$$\left\langle\left\langle \varphi^a_{\mathbf{r}'_1} \varphi^b_{\mathbf{r}'_2} \varphi^c_{\mathbf{r}'_3} \varphi^d_{\mathbf{r}'_4} \right\rangle\right\rangle \simeq G^{aa'}_{\mathbf{r}'_1\mathbf{x}} G^{bb'}_{\mathbf{r}'_2\mathbf{x}} G^{cc'}_{\mathbf{r}'_3\mathbf{y}} G^{dd'}_{\mathbf{r}'_4\mathbf{y}} \left\langle\left\langle \delta\mathcal{Q}^{a'b'}_{\mathbf{x}} \delta\mathcal{Q}^{c'd'}_{\mathbf{y}} \right\rangle\right\rangle. \tag{9}$$

Furthermore, since $(\beta\hat{J})^{-1}_{\mathbf{r}\mathbf{r}'}\hat{G}_{\mathbf{r}'\mathbf{x}} = \delta_{\mathbf{r}\mathbf{x}} - \hat{\mathcal{Q}}_{\mathbf{r}}\hat{G}_{\mathbf{r}\mathbf{x}}$ and the value of $\hat{\mathcal{Q}}$ contains an extra smallness $\sim 1/W$, thus these sections of diagrams can be replaced by delta-functions:

$$\left\langle\!\left\langle \delta n^a_{\mathbf{r}} \delta n^b_{\mathbf{r}} \delta n^c_{\mathbf{r}'} \delta n^d_{\mathbf{r}'} \right\rangle\!\right\rangle \simeq \left\langle\!\left\langle \hat{\mathcal{Q}}^{ab}_{\mathbf{r}} \hat{\mathcal{Q}}^{cd}_{\mathbf{r}'} \right\rangle\!\right\rangle. \tag{10}$$

Finally, the mean-square fluctuations of the polarizability corresponds to the replica component $a = c \neq b = d$, which can be symmetrized as follows:

$$\overline{\langle \delta n_{\mathbf{r}} \delta n_{\mathbf{r}'} \rangle^2} = \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{a\neq b} \left\langle\!\left\langle \hat{\mathcal{Q}}^{ab}_{\mathbf{r}} \hat{\mathcal{Q}}^{ab}_{\mathbf{r}'} \right\rangle\!\right\rangle, \quad \lim_{n\to 0} \frac{1}{n(n-1)} \mathbb{P}^{aa}_{bb} = \frac{3}{2}. \tag{11}$$

2. Derivation of the Ginzburg–Landau functional. In the main text, the following action for two matrix fields was derived:

$$nS[\hat{\mathcal{G}}, \hat{\mathcal{Q}}] = \frac{1}{2}\mathrm{Tr}(\hat{\mathcal{G}}\hat{\mathcal{Q}}) + \frac{1}{2}\mathrm{Tr}\ln(1 + \beta\hat{J}\hat{\mathcal{Q}}) + \beta n \sum_{\mathbf{r}} F_{\mathrm{v}}[\hat{\mathcal{G}}_{\mathbf{r}}], \tag{12}$$

where $F_{\mathrm{v}}[\hat{\mathcal{G}}]$ is a local free energy of a single-cite problem with the following Hamiltonian:

$$-\beta\hat{H}_{\mathrm{v}}[\hat{\mathcal{G}}] = \frac{1}{2}\sum_{ab} \delta n^a (\beta^2 W^2 + \mathcal{G}^{ab}) \delta n^b + \beta\mu \sum_a \delta n^a. \tag{13}$$

In this section we will derive the expansion of this action around the replica-symmetric solution of saddle point equations in the vicinity of the phase transition. Substituting the expansion $\hat{\mathcal{G}} = \hat{\mathcal{G}}_0 + \delta\hat{\mathcal{G}}$ and $\hat{\mathcal{Q}} = \hat{\mathcal{Q}}_0 + \delta\hat{\mathcal{Q}}$, the fluctuations of the second term can be expressed as follows:

$$\frac{1}{2}\delta\mathrm{Tr}\ln(1 + \beta\hat{J}\hat{\mathcal{Q}}) = \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{2k} \mathrm{Tr}(\hat{G}\delta\hat{\mathcal{Q}})^k = \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{2k} \frac{\mathcal{B}_k}{a^{2k-2}} \mathrm{Tr}(\delta\hat{\mathcal{Q}})^k \tag{14}$$

(the latter identity utilizes the replicon condition $\sum_a \delta\mathcal{Q}_{ab} = 0$), with the following notation:

$$\mathcal{B}_k = \int (d\boldsymbol{q}) G_0^k(\boldsymbol{q}) = \frac{\beta U_0}{2} \frac{1}{k-1} \left(\frac{a^2}{\nu_0 T}\right)^{k-1}, \quad k > 1. \tag{15}$$

The fluctuations of the third term read:

$$\beta n \delta F_{\mathrm{v}}[\hat{\mathcal{G}}] = -\sum_{k=2}^{\infty} \frac{1}{2^k k!} Q_{(a_1 b_1)\ldots(a_k b_k)} \delta\mathcal{G}_{a_1 b_1} \ldots \delta\mathcal{G}_{a_k b_k}, \tag{16}$$

where the following irreducible correlation function with independent variables being pairs $\delta n_{a_i}\delta n_{b_i}$ was introduced:

$$Q_{(a_1 b_1)\ldots(a_k b_k)} \equiv \langle\langle (\delta n_{a_1}\delta n_{b_1}) \ldots (\delta n_{a_k}\delta n_{b_k}) \rangle\rangle_{\mathrm{v}}, \tag{17}$$

and the average is performed w.r.t. the Hamiltonian $\hat{H}_{\mathrm{v}}[\hat{\mathcal{G}}_0]$.

The soft mode in this expansion is $\delta\hat{\mathcal{G}} = \hat{\Psi}$ и $\delta\hat{\mathcal{Q}} = Q_{22}\hat{\Psi}$. The term $\mathrm{Tr}\ln$ then reads explicitly:

$$\frac{1}{2}\delta\mathrm{Tr}\ln(1 + \beta\hat{J}\hat{\mathcal{Q}}) = \nu_0 T_c \sum_{k=2}^{\infty} \left(-\frac{1}{6}\right)^{k-1} \frac{1}{2k(k-1)} \mathrm{Tr}\hat{\Psi}^k = \nu_0 T_c \left(-\frac{1}{24}\mathrm{Tr}\hat{\Psi}^2 + \frac{1}{432}\mathrm{Tr}\hat{\Psi}^3 - \frac{1}{5184}\mathrm{Tr}\hat{\Psi}^4 + \ldots\right). \tag{18}$$

On the other hand, the F_{v} term generates terms with different replica structure:

$$\beta n \delta^{(3)} F_{\mathrm{v}}[\hat{\mathcal{G}}] = -\frac{1}{12}\left(Q_{33}\sum_{ab}\Psi^3_{ab} + 2Q_{222}\mathrm{tr}\hat{\Psi}^3\right) = -\nu_0 T \left(\frac{1}{360}\sum_{ab}\Psi^3_{ab} + \frac{1}{180}\mathrm{tr}\hat{\Psi}^3\right), \tag{19}$$

$$\begin{aligned} \beta n \delta^{(4)} F_{\mathrm{v}}[\hat{\mathcal{G}}] = -&\left(\frac{5}{32}Q_{2222}\mathrm{tr}\hat{\Psi}^4 + \frac{1}{48}Q_{44}\sum_{ab}\Psi^4_{ab} + \frac{1}{8}Q_{422}\sum_{abc}\Psi^2_{ab}\Psi^2_{ac} + \frac{1}{4}Q_{332}\sum_{abc}\Psi^2_{ab}\Psi_{ac}\Psi_{bc}\right) = \\ &= -\nu_0 T\left(\frac{1}{896}\mathrm{tr}\hat{\Psi}^4 + \frac{1}{2016}\sum_{ab}\Psi^4_{ab} + \frac{1}{840}\sum_{abc}\Psi^2_{ab}\Psi_{ac}\Psi_{bc} - \frac{1}{840}\sum_{abc}\Psi^2_{ab}\Psi^2_{ac}\right), \end{aligned} \tag{20}$$

where we have denoted:

$$Q_{222}=\int\frac{\nu(u)du}{(2\cosh\frac{\beta(u-\mu)}{2})^6}\approx\frac{\nu_0T}{30},\quad Q_{2222}=\int\frac{\nu(u)du}{(2\cosh\frac{\beta(u-\mu)}{2})^8}\approx\frac{\nu_0T}{140},\tag{21}$$

$$Q_{33}=\int\frac{\nu(u)du\tanh^2\frac{\beta(u-\mu)}{2}}{(2\cosh\frac{\beta(u-\mu)}{2})^4}\approx\frac{\nu_0T}{30},\quad Q_{332}=\int\frac{\nu(u)du\tanh^2\frac{\beta(u-\mu)}{2}}{(2\cosh\frac{\beta(u-\mu)}{2})^6}\approx\frac{\nu_0T}{210},\tag{22}$$

$$Q_{44}=\int\frac{\nu(u)du\left(\left(2\sinh\frac{\beta(u-\mu)}{2}\right)^2-2\right)^2}{\left(2\cosh\frac{\beta(u-\mu)}{2}\right)^8}\approx\frac{\nu_0T}{42},\quad Q_{422}=\int\frac{\nu(u)du\left(\left(2\sinh\frac{\beta(u-\mu)}{2}\right)^2-2\right)}{(2\cosh\frac{\beta(u-\mu)}{2})^8}=-\frac{\nu_0T}{105}.\tag{23}$$

3. One-step replica symmetry breaking. The free energy per lattice cite in the saddle point approximation (neglecting the spatial fluctuations of matrices) contains several terms $\beta F=S[\hat{\mathcal{G}},\hat{\mathcal{Q}}]/N=(S_{\rm L}[\hat{\mathcal{G}},\hat{\mathcal{Q}}]+S_{\rm f}[\hat{\mathcal{Q}}])/N+\beta F_{\rm v}[\hat{\mathcal{G}}]$, which in the one-step replica symmetry breaking (1RSB) scheme read:

$$S_{\rm L}[\hat{\mathcal{G}},\hat{\mathcal{Q}}]/N=\operatorname{tr}(\hat{\mathcal{G}}\hat{\mathcal{Q}})/2n=\frac{1}{2}\left(-\frac{1-m}{m}\mathcal{G}_0\mathcal{Q}_0+\frac{1}{m}\mathcal{G}_0\mathcal{Q}_1+\mathcal{G}_0\mathcal{Q}_2+\mathcal{G}_1\mathcal{Q}_1+m\mathcal{G}_1\mathcal{Q}_2+\mathcal{G}_2\mathcal{Q}_1\right),\tag{24}$$

$$\begin{aligned}S_{\rm f}/N=\operatorname{Tr}\ln(1+\beta\hat{J}\hat{\mathcal{Q}})/2Nn=\frac{\beta U_0}{4}\left(-\left(\frac{1}{m}-1\right)\left(\mathcal{Q}_0+\mathcal{Q}_0\ln\frac{1}{\beta U_0\mathcal{Q}_0}\right)+\right.\\ \left.+\frac{1}{m}\left(\mathcal{Q}_1+\mathcal{Q}_1\ln\frac{1}{\beta U_0\mathcal{Q}_1}\right)+\mathcal{Q}_2\ln\frac{1}{\beta U_0\mathcal{Q}_1}\right),\end{aligned}\tag{25}$$

$$\beta F_{\rm v}=\frac{1}{2}(\mathcal{G}_0+m\mathcal{G}_1)\left(\frac{1}{2}-K\right)^2-\frac{1}{8}\mathcal{G}_0-\beta\widetilde{\mu}\left(\frac{1}{2}-K\right)-\frac{1}{m}\int du_2\nu_2(u_2)\ln\Xi(u_2),\tag{26}$$

where we have introduced renormalized chemical potential $\widetilde{\mu}=\mu+T\left(\mathcal{G}_0+m\mathcal{G}_1\right)\left(\frac{1}{2}-K\right)$, renormalized disorder strength $\widetilde{W}=\sqrt{W^2+T^2\mathcal{G}_2}$, and two auxiliary "distribution functions":

$$\nu_2(u_2)=\frac{\exp(-u_2^2/2\widetilde{W}^2)}{\sqrt{2\pi}\widetilde{W}},\quad \nu_1(u_1,u_2)=\frac{\exp(-u_1^2/2T^2\mathcal{G}_1)}{\sqrt{2\pi\mathcal{G}_1}T}\left[2\cosh\frac{\beta(u_1+u_2-\widetilde{\mu})}{2}\right]^m,\quad \Xi(u_2)=\int du_1\nu_1(u_1,u_2).\tag{27}$$

One can extract the leading asymptotic behavior of the integral over u_2 making use of the small parameter $U_0/W\ll 1$:

$$\beta F_{\rm v}=\frac{1}{2}(\mathcal{G}_0+m\mathcal{G}_1)K(1-K)-\beta\widetilde{\mu}\left(\frac{1}{2}-K\right)-\left\langle\ln\left(2\cosh\frac{\beta(u_2-\widetilde{\mu})}{2}\right)\right\rangle_2-\frac{1}{2}\nu_0Tf_{\rm v}(m,\mathcal{G}_1),\tag{28}$$

where the following dimensionless function was introduced:

$$f_{\rm v}(m,\mathcal{G}_1)=\frac{2}{m}\int dz\left(\ln\Xi(z,m,\mathcal{G}_1)-m\ln 2\cosh\frac{z}{2}-\frac{m^2\mathcal{G}_1}{8}\right),\tag{29}$$

$$\Xi(z,m,\mathcal{G}_1)=\Xi\left(u_2\equiv\widetilde{\mu}+Tz\right)=\int\frac{dye^{-y^2/2\mathcal{G}_1}}{\sqrt{2\pi\mathcal{G}_1}}\left[2\cosh\frac{y+z}{2}\right]^m,\tag{30}$$

with the variables $z=\beta(u_2-\widetilde{\mu})$, and $y=\beta u_1$. The variation of the full free energy w.r.t. $\mathcal{Q}_i$ and $\mathcal{G}_i$ yield equations for $\mathcal{G}_i$ and $\mathcal{Q}_i$ respectively — to Eqs. (35), (36).

3.1. Analysis of the equations in the $T\ll T_c$ limit. The solution in the low temperature limit behaves as $m\ll 1$, $\mathcal{G}_1\gg 1$, $\xi\equiv m^2\mathcal{G}_1/8=O(1)$. Such scaling allows us to calculate:

$$\begin{aligned}\Xi\left(z=\frac{x}{m},m,\mathcal{G}_1\right)\underset{m\ll1}{\equiv}\Xi(x,\xi)=\int\frac{dy}{4\sqrt{\pi\xi}}\exp\left(-\frac{y^2}{16\xi}+\frac{1}{2}|y+x|\right)=\\=\frac{e^{\xi}}{2}\left[e^{y/2}\left(1+\operatorname{erf}\left(\frac{4\xi+x}{4\sqrt{\xi}}\right)\right)+e^{-x/2}\left(1+\operatorname{erf}\left(\frac{4\xi-x}{4\sqrt{\xi}}\right)\right)\right],\end{aligned}\tag{31}$$

while for auxiliary dimensionless function the following scaling holds: $f_{\mathrm{v}}(m, \mathcal{G}_1) = 8f(\xi)/m^2$, where:

$$f(\xi) = \frac{1}{4}\int dx \left(\ln \Xi(x,\xi) - \frac{|x|}{2} - \xi\right). \tag{32}$$

The saddle point equations for $q \equiv \mathcal{Q}_0/\nu_0 T$, ξ и m can be written as follows:

$$\begin{cases} q & = f'(\xi)/(1-m) \\ \xi & = \frac{3}{4}m\beta T_c \ln \frac{1}{f'(\xi)} \\ 2f(\xi) - \xi f'(\xi) & = \frac{3}{4}m\beta T_c(1-q) \end{cases} \tag{33}$$

Assuming the scaling $m = \mu(T/T_c)$, $\mu = O(1)$, the system of equations becomes fully dimensionless, and can be reduced to the single equation for ξ variable, which can then be solved numerically:

$$\xi = \frac{2f(\xi) - \xi f'(\xi)}{1 - f'(\xi)} \ln \frac{1}{f'(\xi)} \Rightarrow \xi \approx 9.17, \tag{34}$$

$$q = f'(\xi) \approx 1.43 \cdot 10^{-5}, \quad \mu = \frac{4(2f(\xi) - \xi f'(\xi))}{3(1-q)} \approx 1.10. \tag{35}$$

Due to the large value of ξ, these numerical solutions can be obtained analytically with good precision. The scaling function has the following asymptotic behavior:

$$f(\xi) \approx \frac{\pi^2}{24} - \frac{1}{4}\sqrt{\frac{\pi}{\xi}}e^{-\xi}, \quad \xi \gg 1 \tag{36}$$

so that $q \approx \frac{1}{4}\sqrt{\pi/\xi}e^{-\xi}$ (which yields $1.52 \cdot 10^{-5}$), and $\mu \approx 8f(\xi)/3 \approx \pi^2/9$ (which yields 1.10). Substituting also this asymptotic to the equation for ξ, one can see that it does contain numerically small parameter $\epsilon = \frac{6}{\pi^2} - \frac{1}{2} \approx 0.11$ and has the approximate form $\ln \frac{16\xi}{\pi} \approx \epsilon\xi$.

3.2. Distribution function of the local pinning potential. The distribution function of the vortex local pinning potential is defined as follows:

$$P(u) = \langle\langle \delta(u - (u_1 + u_2))\rangle_1\rangle_2 \equiv \int du_2 \nu_2(u_2)\frac{1}{\Xi(u_2)}\int du_1 \nu_1(u_1, u_2)\delta(u - (u_1 + u_2)), \tag{37}$$

where the averages $\langle \dots \rangle_1$ and $\langle \dots \rangle_2$ are taken w.r.t. distribution functions defined in (27).

At low temperatures, the distribution function is noticeably modified in the vicinity of the chemical potential in the region of size $\propto T_c$. We will then calculate the distribution function of the rescaled variable $h \equiv (u - \widetilde{\mu})/T_c$. The asymptotic behavior of the function $\Xi(u_2)$ was already obtained above, see Eq. (31), where $z = \beta(u_2 - \widetilde{\mu})$. In this limit, the distribution function reads:

$$P(h) = \nu_0 T_c \cdot \frac{\exp(\mu|h|/2)}{4\sqrt{\pi\xi}} \int \frac{dx}{\Xi(x,\xi)} \exp\left(-\frac{(\mu h - x)^2}{16\xi}\right). \tag{38}$$

Just like in the previous section, these expression can be further simplified analytically for $\xi \gg 1$, and read as follows:

$$\Xi(x,\xi) \approx \exp(|x|/2 + \xi), \quad P(h) \approx \nu_0 T_c \cdot \frac{1}{2}\mathrm{erfc}\left(\frac{4\xi - \mu|h|}{4\sqrt{\xi}}\right). \tag{39}$$

3.3. Low temperature behavior of the entropy. The expression for the entropy can be obtained by differentiating the full free energy w.r.t. T. If one also takes into account the saddle point equations, one obtains the following simple expression valid for arbitrary T:

$$S = \nu_0 T\left(f_{\mathrm{v}}(m, \mathcal{G}_1) + \frac{1}{2}m\frac{\partial f_{\mathrm{v}}}{\partial m} - \mathcal{G}_1\frac{\partial f_{\mathrm{v}}}{\partial \mathcal{G}_1} + \frac{\pi^2}{3}\right) - 3\beta T_c \mathcal{Q}_0. \tag{40}$$

As we have shown above, in the low temperature limit auxiliary function f_{v} satisfies scaling relation $f_{\mathrm{v}}(m, \mathcal{G}_1) \approx 8f_{\mathrm{v}}(\xi)/m^2$. This scaling relation nullifies the combination of first three terms in the equation above.

However, since $f_{\rm v} \propto \beta^2$, such cancellation only guarantees the absence of the unphysical terms $\sim 1/T$ in the entropy. In order to extract the low-temperature behavior of the entropy, one should consider corrections to this scaling:

$$\Delta f_{\rm v}(m,\mathcal{G}_1) \equiv f_{\rm v}(m,\mathcal{G}_1) - \frac{8}{m^2} f(\xi) = \frac{2}{m^2}\int dx \ln \frac{\Xi(\frac{x}{m},m,\mathcal{G}_1)}{\Xi(x,\xi)} - \frac{\pi^2}{3}. \tag{41}$$

The quantity under the logarithm is close to unity when $m \ll 1$, which allows us to expand:

$$\Delta f_{\rm v}(m,\mathcal{G}_1) = \frac{2}{m^2}\int \frac{dx}{\Xi(x,\xi)} \int \frac{dy}{4\sqrt{\pi\xi}} \exp\left(-\frac{y^2}{16\xi}\right)\left(\left[2\cosh\frac{y+x}{2m}\right]^m - e^{|x+y|/2}\right) - \frac{\pi^2}{3} \underset{m\ll 1}{\approx} \frac{\pi^2}{3}(g(\xi)-1) \tag{42}$$

with:

$$g(\xi) = \frac{1}{4\sqrt{\pi\xi}}\int \frac{dx}{\Xi(x,\xi)} \exp\left(-\frac{x^2}{16\xi}\right) \equiv \frac{P(h=0)}{\nu_0 T_c}. \tag{43}$$

The low-temperature entropy then reads $S = -3\beta T_c \mathcal{Q}_0 + \frac{\pi^2}{3}\nu_0 T g(\xi) \to -3\beta T_c \mathcal{Q}_0$.