Supplementary Material to the article "Local quench within Keldysh technique"

1. INTRODUCTION

The aim of this supplementary material is to demonstrate that the new formalism developed in the main article is in agreement with the previous calculations performed in the framework of the conformal field theory. Let us consider the example of a free scalar field, which can be described by the conformal field theory with the central charge $c = 1$. In order to perform these calculations, we use field mode expansion (mass m is finite for the intermediate calculations, but it is set to zero at the end).

$$\hat{\varphi}(x) = \int \frac{dp}{2\pi} \frac{1}{\sqrt{2\omega_p}} \left(\hat{a}_p e^{ipx} + \hat{a}_p^\dagger e^{-ipx}\right) \equiv \hat{\phi}(x) + \hat{\phi}^\dagger(x). \tag{S1}$$

Here, we define positive and negative frequency parts. The former contains only field annihilation operators; the later - only creation operators. Creation and annihilation operators have the following canonical commutation relations: $[\hat{a}_p, \hat{a}_q^\dagger] = 2\pi\delta(p-q)$. As an initial condition before a quench we consider the vacuum state $|0\rangle$, which is destroyed by any annihilation operator $\hat{a}_p|0\rangle = 0$, which means that $\hat{\phi}(x)|0\rangle = 0$. After a local quench the state became (see, for example [1, 2, 3, 4, 5]):

$$|\psi_0\rangle = \mathcal{N} e^{-\epsilon \hat{H}} \hat{Q}(x_q)|0\rangle. \tag{S2}$$

In the above expression, a small ϵ acts as the UV(ultraviolet) regularisator, $\mathcal{N}$ – normalisation constant, such as $\langle\psi_0|\psi_0\rangle = 1$.

The Hamiltonian of a free scalar theory is:

$$\hat{H} = \int \frac{dp}{2\pi} \omega_p \hat{a}_p^\dagger \hat{a}_p. \tag{S3}$$

In the framework of the conformal field theory, it is more convenient if $\hat{Q}(x)$ is a primary operator. As an example of such an operator, we consider the vertex operator $\hat{Q}(x) = \hat{\mathcal{V}}_\alpha(x) =: e^{i\alpha\hat{\varphi}(x)} :$, where with $: ... :$ the normal ordering is denoted. Moreover, as a second example, we consider the quench operator $\hat{Q}(x) = \hat{\varphi}(x)$, which is not a primary operator; however, it was thoroughly investigated in the paper[1].

2. VERTEX OPERATOR QUENCH

Let us consider a state that is created after a local quench by the vertex operator $\hat{\mathcal{V}}_\alpha(x)$:

$$\begin{aligned}|\psi_0\rangle = \mathcal{N} e^{-\epsilon\hat{H}} : e^{i\alpha\hat{\varphi}(x_q)} : |0\rangle = \mathcal{N} e^{-\epsilon\hat{H}} e^{i\alpha\hat{\phi}^\dagger(x_q)} e^{i\alpha\hat{\phi}(x_q)}|0\rangle = \mathcal{N} e^{-\epsilon\hat{H}} e^{i\alpha\hat{\phi}^\dagger(x_q)}|0\rangle \\ = \mathcal{N} e^{-\epsilon\hat{H}} e^{i\alpha\hat{\phi}^\dagger(x_q)} e^{\epsilon\hat{H}}|0\rangle = \mathcal{N} e^{i\alpha e^{-\epsilon\hat{H}}\hat{\phi}^\dagger(x_q) e^{\epsilon\hat{H}}}|0\rangle.\end{aligned} \tag{S4}$$

With the help of commutation relation $e^{-\epsilon\hat{H}} \hat{a}_p^\dagger e^{\epsilon\hat{H}} = \hat{a}_p^\dagger e^{-\epsilon\omega_p}$ one can show that

$$e^{-\epsilon\hat{H}} \hat{\phi}^\dagger(x_q) e^{\epsilon\hat{H}} = \int \frac{dp}{2\pi} \frac{1}{\sqrt{2\omega_p}} \hat{a}_p^\dagger e^{-ipx_q - \epsilon\omega_p}. \tag{S5}$$

Let us introduce the "smearing" function:

$$\eta(x) = \int \frac{dp}{2\pi} e^{ipx - \epsilon\omega_p} = \frac{m\epsilon}{\pi\sqrt{x^2+\epsilon^2}} K_1\left(m\sqrt{x^2+\epsilon^2}\right), \tag{S6}$$

here $K_\nu(z)$ is the Macdonald function. With the help of this function, previous relation can be rewritten as:

$$e^{-\epsilon\hat{H}} \hat{\phi}^\dagger(x_q) e^{\epsilon\hat{H}} = \int dx\ \eta(x - x_q)\phi^\dagger(x) \equiv \hat{\phi}_s^\dagger(x_q). \tag{S7}$$

In other words, for the problem under consideration, the action of the regularisation multiplier $e^{-\epsilon\hat{H}}$ can be mimicked by the introduction of a certain "smearing" function (S6). Than for the equation (S4) we have

$$|\psi_0\rangle = \mathcal{N}e^{i\alpha e^{-\epsilon\hat{H}}\hat{\phi}^\dagger(x_q)e^{\epsilon\hat{H}}}|0\rangle = \mathcal{N}e^{i\alpha\hat{\phi}_s^\dagger(x_q)}|0\rangle = \mathcal{N}e^{i\alpha\hat{\phi}_s^\dagger(x_q)}e^{i\alpha\hat{\phi}_s(x_q)}|0\rangle = e^{i\alpha(\hat{\phi}_s^\dagger(x_q)+\hat{\phi}_s(x_q))}|0\rangle = e^{i\alpha\hat{\varphi}_s(x_q)}|0\rangle. \tag{S8}$$

In the above equation, we use the fact that the commutator of $\hat{\phi}_s(x)$ and $\hat{\phi}_s^\dagger(x)$ is a number, which can be combined with the normalisation constant $\mathcal{N}$. Therefore, we have shown that the problem of the quench induced by the vertex operator can be considered as a particular case of the problem solved in the main article. Namely, it is the case of the potential $V(\varphi) = -\varphi$ and vacuum initial state $T = 0$ ($f_p = 0$). In this case, from the expression (36) of the main article, it follows that

$$\langle\hat{\varepsilon}\rangle_t^Q = \langle\hat{\varepsilon}\rangle_t + \frac{1}{2}\alpha^2\Big(m^2\big(G_R^s(t,x)\big)^2 + \big(\partial_t G_R^s(t,x)\big)^2 + \big(\partial_x G_R^s(t,x)\big)^2\Big). \tag{S9}$$

If the "smearing" function has the form of (S6) than all necessary integrals can be done exactly. In particular, the "smeared" retarded Green function is:

$$G_R^s(t,x) = \int dy\ \eta(y-x_q)G_R(t,x-y) = -\theta(t)\int\frac{dp}{2\pi}\frac{\sin(\omega_p t)}{\omega_p}e^{-ip(x-x_q)-\epsilon\omega_p}$$
$$= \frac{i}{2\pi}\theta(t)\left(K_0\left(m\sqrt{(x-x_q)^2+(\epsilon-it)^2}\right) - K_0\left(m\sqrt{(x-x_q)^2+(\epsilon+it)^2}\right)\right). \tag{S10}$$

The answer looks especially simple in the massless case $m \to 0$:

$$G_R^s(t,x) = \frac{i}{4\pi}\log\left(\frac{(x-x_q)^2+(\epsilon+it)^2}{(x-x_q)^2+(\epsilon-it)^2}\right). \tag{S11}$$

Than the energy density is:

$$\langle\hat{\varepsilon}\rangle_t^Q = \langle\hat{\varepsilon}\rangle_t + \frac{\alpha^2}{4\pi^2}\left(\frac{\epsilon^2}{((x-x_q-t)^2+\epsilon^2)^2} + \frac{\epsilon^2}{((x-x_q+t)^2+\epsilon^2)^2}\right). \tag{S12}$$

Now we can compare the above answer with the results of the conformal field theory. In order to do this, we need to turn to imaginary time $\tau = it$ and introduce the complex coordinates $z = x + i\tau$, $\bar{z} = x - i\tau$. Quench is done by the operator $\hat{Q}(z,\bar{z})$ with the conformal dimensions $(h,\bar{h})$. In order to find the energy density, it is necessary to calculate the following ratio of the eucleadean correlation functions [1, 4]:

$$\delta\varepsilon(\tau,x) = -\frac{1}{2\pi}\frac{\langle Q^\dagger(x_q+i\epsilon,x_q-i\epsilon)(T(x+i\tau)+\bar{T}(x-i\tau))Q(x_q-i\epsilon,x_q+i\epsilon)\rangle_E}{\langle Q^\dagger(x_q+i\epsilon,x_q-i\epsilon)Q(x_q-i\epsilon,x_q+i\epsilon)\rangle_E}, \tag{S13}$$

and perform analytical continuation $\tau \to it$. Here $T(z)$ and $\bar{T}(\bar{z})$ are holomorphic and antiholomorphic components of the stress-energy tensor, and a common multiplier is chosen to correspond to the energy density we calculated above. The correlation function in the numerator can be reduced to a pair correlation function using the conformal Ward identity [6], and the energy density become [1, 4]:

$$\delta\varepsilon(\tau,x) = \frac{2h\epsilon^2}{\pi(x_q-x-i\epsilon-i\tau)^2(x_q-x+i\epsilon-i\tau)^2} + \frac{2\bar{h}\epsilon^2}{\pi(x_q-x+i\epsilon+i\tau)^2(x_q-x-i\epsilon+i\tau)^2}. \tag{S14}$$

After that, we need to perform an analytical continuation and use the fact that the conformal dimensions of the vertex operator $\hat{\mathcal{V}}_\alpha =: e^{i\alpha\hat{\varphi}}:$ equal to $(\alpha^2/(8\pi), \alpha^2/(8\pi))$ [6]. As a result, we obtain exactly the answer of the expression (S12).

3. QUENCH WITH THE OPERATOR $\hat{\varphi}(\mathrm{x_q})$

Quench with the operator $\hat{Q}(x) = \hat{\varphi}(x)$ has been considered in the work [1]. Although the expression (36) from the main article is not directly applicable to this case, the developed method can be easily generalised to treat such a quench. In this case, the equation (20) of the main article

$$\langle\hat{O}\rangle_t^Q = \int\mathfrak{D}\Phi(\mathbf{x})\mathfrak{D}\Pi(\mathbf{x})W[\Phi(\mathbf{x}),\Pi(\mathbf{x})]\,\mathrm{Q}\Big(\Phi_\mathrm{s}, -\frac{\delta}{\delta\Pi_\mathrm{s}}\Big)\,O[\phi_c(t,\Phi(\mathbf{x}),\Pi(\mathbf{x}))] \tag{S15}$$

is still correct if

$$\mathrm{Q}\Big(\Phi_{\mathrm{s}},\text{-}\frac{\delta}{\delta\Pi_{\mathrm{s}}}\Big) = \mathcal{N}^2\left(\Phi_s + i\frac{\hbar}{2}\frac{\delta}{\delta\Pi_s}\right)\left(\P hi_s - i\frac{\hbar}{2}\frac{\delta}{\delta\Pi_s}\right) = \mathcal{N}^2\left(\Phi_s^2 + \frac{\hbar^2}{4}\frac{\delta^2}{\delta\Pi_s^2}\right). \tag{S16}$$

Here normalisation constant $\mathcal{N}$ is defined from the condition $\langle 1\rangle_t^Q = 1$:

$$\mathcal{N}^{-2} = \int \mathfrak{D}\Phi(\mathbf{x})\mathfrak{D}\Pi(\mathbf{x})W[\Phi(\mathbf{x}),\Pi(\mathbf{x})]\Phi_s^2 = \langle\Phi_s^2\rangle_{i.c.}. \tag{S17}$$

With the above condition, further calculations are performed analogously to those in the main article. The energy density after quench equals:

$$\langle\hat{\varepsilon}\rangle_t^Q = \frac{1}{2\langle\Phi_s^2\rangle_{i.c.}}\langle\Phi_s^2(m^2\phi_c^2(t,x) + (\partial_t\phi_c(t,x))^2 + (\partial_x\phi_c(t,x))^2)\rangle_{i.c.} + \frac{\hbar^2}{4\langle\Phi_s^2\rangle_{i.c.}}\Big(m^2\big(G_R^s(t,x)\big)^2 + \big(\partial_t G_R^s(t,x)\big)^2 + \big(\partial_x G_R^s(t,x)\big)^2\Big). \tag{S18}$$

If we use vacuum initial conditions, then the Wick theorem is applicable to the calculation of the averages. The result is:

$$\langle\hat{\varepsilon}\rangle_t^Q = \langle\hat{\varepsilon}\rangle_t^Q - \frac{1}{\langle\Phi_s^2\rangle_{i.c.}}\Big(m^2\big(G_K^s(t,x)\big)^2 + \big(\partial_t G_K^s(t,x)\big)^2 + \big(\partial_x G_K^s(t,x)\big)^2\Big) + \frac{\hbar^2}{4\langle\Phi_s^2\rangle_{i.c.}}\Big(m^2\big(G_R^s(t,x)\big)^2 + \big(\partial_t G_R^s(t,x)\big)^2 + \big(\partial_x G_R^s(t,x)\big)^2\Big). \tag{S19}$$

As in the previous section, for the vacuum initial state the calculations can be done analytically. In the case of reqularisation used in the work [1], the "smearing" function should be chosen in the form (S6). So, the energy density is:

$$\langle\Phi_s^2\rangle_{i.c.} = i\int dydz\;\eta(y-x_q)\eta(z-x_q)G_K^s(0,y-z) = \hbar\int\frac{dp}{2\pi}\frac{1}{2\omega_p}e^{-2\epsilon\omega_p} = \frac{\hbar}{2\pi}K_0(2m\epsilon), \tag{S20}$$

$$iG_K^s(t,x) + i\frac{\hbar}{2}G_R^s(t,x) = \hbar\int\frac{dp}{2\pi}\frac{1}{2\omega_p}e^{-ip(x-x_q)-it\omega_p-\epsilon\omega_p} = \frac{\hbar}{2\pi}K_0\left(m\sqrt{(x-x_q)^2+(\epsilon+it)^2}\right), \tag{S21}$$

$$\langle\hat{\varepsilon}\rangle_t^Q = \langle\hat{\varepsilon}\rangle_t^Q + \frac{m^2\hbar}{2\pi K_0(2m\epsilon)}\left|K_0\left(m\sqrt{(x-x_q)^2+(\epsilon+it)^2}\right)\right|^2 + \frac{m^2\hbar\left((x-x_q)^2+t^2+\epsilon^2\right)}{2\pi K_0(2m\epsilon)}\left|\frac{K_1\left(m\sqrt{(x-x_q)^2+(\epsilon+it)^2}\right)}{\sqrt{(x-x_q)^2+(\epsilon+it)^2}}\right|^2. \tag{S22}$$

Exactly the same answer was obtained in the work [1] with the help of eucleadean time calculations and analytical continuation. Therefore, the results obtained for the vacuum initial state can be calculated using the method of the main article with the correct choice of the "smearing" function $\eta(x)$. It is important to note that the presented method is more general and can be applied to different quench operators and arbitrary initial conditions.

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