Supplementary Material to the article "Correlation functions of passive scalar as a measure of velocity gradient statistics"

A. RELATION BETWEEN F_4 AND P IN THREE-DIMENSIONAL CASE

The dynamics of $\hat{I}$ (13) coincides with the dynamics of $\hat{\mathcal{R}} = \hat{W}\big(\hat{\mathbf{r}}_1(0)\mathbf{r}_1^T(0) + \mathbf{r}_2(0)\mathbf{r}_2^T(0)\big)W^T$, so $\hat{I} = \hat{W}\hat{W}^T$ and $\mu_i = \rho_i$ [1]. We define the unit eigenvectors $\mathbf{n}$, $\boldsymbol{\ell}$, $\mathbf{m}$ of the matrix $\hat{\mathcal{R}} = \mathbf{r}_1\mathbf{r}_1^T + \mathbf{r}_2\mathbf{r}_2^T$ as

$$\hat{\mathcal{R}}\mathbf{n} = r^2\mathbf{n}, \qquad \hat{\mathcal{R}}\boldsymbol{\ell} = r_\perp^2\boldsymbol{\ell}, \qquad \mathbf{m} = [\mathbf{n}\times\boldsymbol{\ell}]. \tag{S1}$$

Consider equations governing the dynamics of matrices $\hat{\mathcal{O}} = (\mathbf{n}, \boldsymbol{\ell}, \mathbf{m})$ and $\hat{\mathcal{M}}$ defined in (14) in the limit of strong deformation of fluid element, $e^{\rho} \gg e^{\rho_2} \gg e^{\rho_3}$. Let

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{O} = \mathcal{O}\hat{\Omega} \;=\; \hat{\omega}\hat{\mathcal{O}}, \quad \boldsymbol{\omega} = \hat{\mathcal{O}}\boldsymbol{\Omega}, \quad \omega_i = -\frac{1}{2}\epsilon_{ijk}\omega_{jk}. \tag{S2}$$

The identity $\hat{\omega}\mathbf{r} = [\boldsymbol{\omega}\times\mathbf{r}]$ follows from the given definitions. The equation (13) is equivalent to

$$\hat{\Omega}\hat{\mathcal{M}} - \hat{\mathcal{M}}\hat{\Omega} + \frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathcal{M}} = \widetilde{\sigma}\hat{\mathcal{M}} + \hat{\mathcal{M}}\widetilde{\sigma}^T, \qquad \widetilde{\sigma} \;=\; \hat{\mathcal{O}}^T\hat{\sigma}\hat{\mathcal{O}}. \tag{S3}$$

Diagonal elements of (S3) read

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_{ii} = 2\widetilde{\sigma}_{ii}\mathcal{M}_{ii}: \qquad \begin{cases} \dot{\rho} = \sigma_{ik}n_in_k, \\ \dot{\rho}_2 = \sigma_{ik}m_im_k. \end{cases} \tag{S4}$$

Next, we should keep only the first summands in both sides of (S3), if the first index is greater than the second one, $i > k$, so $\Omega_{ik} = \widetilde{\sigma}_{ik}$. This means

$$\Omega_1 = \sigma_{ik}m_i\ell_k, \quad \Omega_2 = -\sigma_{ik}m_in_k, \quad \Omega_3 = \sigma_{ik}\ell_in_k, \tag{S5}$$

in terms of unit vectors (S1). Equivalently, the angular velocity of $\mathbf{r}_{1,2}$ rotation is

$$\boldsymbol{\omega} = m_i\sigma_{ik}\ell_k\mathbf{n} - m_i\sigma_{ik}n_k\boldsymbol{\ell} + \ell_i\sigma_{ik}n_k\mathbf{m}. \tag{S6}$$

The equation $\dot{\mathbf{n}} = [\boldsymbol{\omega}\times\mathbf{n}]$ coincides with the equation $\dot{n}_i = \delta^{\perp}_{il}\sigma_{lk}n_k$ given in the main text, and we further need in the next equation $\dot{\boldsymbol{\ell}} = [\boldsymbol{\omega}\times\boldsymbol{\ell}] = \mathbf{m}\, m_i\sigma_{ik}\ell_k - \ell_i\sigma_{ik}n_k\mathbf{n}$.

Let us consider the relatively simple transition from the equation (8) to the equation (6) before the deriving the dynamical equation for joint probability distribution function for ρ, ρ_2. After the substitution of $\dot{\rho}$ and $\dot{n}_i$, the equation (8) becomes

$$\partial_t\big(\mathcal{P}e^{-d\rho}\big) = -\partial_\rho\big(n_k\sigma_{kl}n_l\,\mathcal{P}e^{-d\rho}\big) - \delta^{\perp}_{ik}\sigma_{kl}r_l\frac{\partial}{\partial r_i}\big(\mathcal{P}e^{-d\rho}\big). \tag{S7}$$

Taking into account that $\partial_\rho = rn_i\partial_i$ and $rn_l = r_l$, we arrive to the equation (6) with $\mathcal{F} = \mathcal{P}e^{-d\rho}$. Note that equality (10) could be obtained only using averaging over angles but without further transformation of (S7), which is more cumbersome. Now we proceed with technically more complicated equation for $\mathcal{P}(t, \rho, \rho_2, \hat{\mathcal{O}})$. Let us consider unit mutually orthogonal vectors $\mathbf{n}$ and $\boldsymbol{\ell}$ to be formally independent. Deriving an analog of (8), it is necessary to exclude differentiation not only in the directions of the increase in the amplitudes of the vectors, but also in the direction of change in their scalar product $(\mathbf{n}\cdot\mathbf{m}) = 0$. These restrictions lead to the following equation:

$$\begin{aligned} \partial_t\mathcal{P} \;=\; & -\partial_\rho\big(\dot{\rho}\mathcal{P}\big) - \partial_{\rho_2}\big(\dot{\rho}_2\mathcal{P}\big) - \\ & - \left(m_im_k\frac{\partial}{\partial n_k} + \frac{\ell_i}{2}\left(\ell_k\frac{\partial}{\partial n_k} - n_k\frac{\partial}{\partial \ell_k}\right)\right)\big(\dot{n}_i\,\mathcal{P}\big) - \\ & - \left(m_im_k\frac{\partial}{\partial \ell_k} + \frac{n_i}{2}\left(n_k\frac{\partial}{\partial \ell_k} - \ell_k\frac{\partial}{\partial n_k}\right)\right)\big(\dot{\ell}_i\,\mathcal{P}\big). \end{aligned} \tag{S8}$$

We transform angular summands in (S8) taking into account that the thirds unit vector $\mathbf{m}$ is given in (S1):

$$\begin{aligned} & \big(4n_i\sigma_{ik}n_k + 2\ell_i\sigma_{ik}\ell_k\big)\mathcal{P} - \\ & -\Bigg(m_i\sigma_{ik}n_km_n\frac{\partial}{\partial n_n} + m_i\sigma_{ik}\ell_km_n\frac{\partial}{\partial \ell_n} + \\ & + \ell_i\sigma_{ik}n_k\left(\ell_n\frac{\partial}{\partial n_n} - n_n\frac{\partial}{\partial \ell_n}\right)\Bigg)\mathcal{P}. \end{aligned} \tag{S9}$$

Directly deriving (17) from equation (S8) would be technically quite complex. Note that, firstly, the parentheses in (S9) contain the dot product of the angular velocity of the rotation (S6) and the rotation generator, and secondly, the partial derivatives can be shifted and placed in front of all other factors without changing the result. So we average the equation (S8) over angles taking into account (S9) and obtain

$$\begin{aligned} \partial_t\big\langle\mathcal{P}e^{-4\rho-2\rho_2}\big\rangle_{ang} \;=\; & -\partial_\rho\big\langle\dot{\rho}\,\mathcal{P}e^{-4\rho-2\rho_2}\big\rangle_{ang} - \\ & -\partial_{\rho_2}\big\langle\dot{\rho}_2\,\mathcal{P}e^{-4\rho-2\rho_2}\big\rangle_{ang}. \end{aligned} \tag{S10}$$

Here, the coefficients in the exponents coincide with the power indexes in (18), which proves (18).

B. SOLUTION OF FOKKER-PLANK EQUATION FOR F_4 IN 2D-CASE

Here we suppose that the velocity gradient is shortly correlated in time in a Lagrangian reference system. General form of the equation for the fourth-order correlation function F_4 in two-dimensional case is established in [2]:

$$\begin{aligned}\partial_t F_4 \; &= \frac{\lambda \sin^2\theta}{2}\Big(\big(2\partial_\theta - \operatorname{ctg}\theta(r_1\partial_1 + r_2\partial_2)\big)^2 + \\ &+ (r_1\partial_1 - r_2\partial_2)^2\Big)F_4 + 2\kappa\big(\Delta_1+\Delta_2\big)F_4 \end{aligned} \tag{S11}$$

in terms of variables r_1, r_2, θ, where vectors $\mathbf{r}_{1,2}$ are introduced before (17) and θ is angle between the vectors. Assuming that F_4 depends only on $A = rr_\perp/l^2$ and $J = (r^2 + r_\perp^2)/2l^2 \geq A$, the equation (S11) becomes

$$\begin{aligned}\frac{1}{2}\partial_\tau F_4 \; &= \partial_J\big(J^2 - A^2\big)\partial_J F_4 + \\ &+ \frac{1}{\mathrm{Pe}}\Big(\partial_A\big(J\partial_A + A\partial_J\big) + \partial_J\big(A\partial_A + J\partial_J\big)\Big)F_4. \end{aligned} \tag{S12}$$

The right-hand side of the equation (S12) is a full divergence, so we call F_4 probability density as well. When Péclet number $\mathrm{Pe} \to \infty$, the dynamics in A vanished in (S12). This corresponds to surface area conservation for fluid contour in a two-dimensional incompressible flow. The boundary conditions for equation (S12) are the normal probability flux at the boundaries $J = A > 0$ and $A = 0$ is zero,

$$\begin{aligned}(J-A)\big((J+A)\partial_J + \mathrm{Pe}^{-1}(\partial_A - \partial_J)\big)F_4\big|_{J=A} \; &= 0, \\ \big(J\partial_A + A\partial_J\big)F_4\big|_{A=0} \; &= 0, \end{aligned} \tag{S13}$$

and the decrease of F_4 at large values A, J is fast enough for the total probability to be finite.

The definition domain is shown in Figure S1. Let us first consider the diffusionless limit for the equation (S12) and adopt $A = 0$, which corresponds to completely collinear geometry with $r_\perp = 0$. Then the general equation (S12) is reduced to

$$\frac{\partial_\tau F_4}{2} = \partial_J J^2 \partial_J F_4, \quad F_4|_{t=0} = e^{-J}, \quad J\partial_J F_4\big|_{J=0} = 0. \tag{S14}$$

Let $\ln J = 2a$ by definition, which corresponds to the limit $r \gg r_\perp$ under consideration. Equation (S14) can be recast as

$$\partial_\tau F_4 = \left(\frac{1}{2}\partial_a^2 + \partial_a\right)F_4, \qquad F_4\big|_{t=0} = \exp\big(-e^{2a}\big), \tag{S15}$$

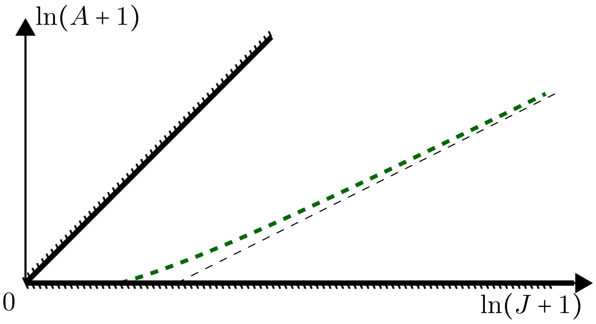

Fig. S1. Ranges of asymptotic behavior of fourth-order correlation function on the J-A plane. Time τ satisfies the inequalities $\ln \mathrm{Pe} < \tau \ll \big(\ln \mathrm{Pe}\big)^2$.

which solution at large times is

$$F_4 = \frac{1}{2^{5/4}\tau^{1/4}} \operatorname{Erfc}\left(\frac{\tau + a}{\sqrt{2\tau}}\right). \tag{S16}$$

The error function $\operatorname{Erfc}(\chi) = e^{-\chi^2}/\sqrt{\pi\chi}$ at large arguments $\chi \gg 1$, so

$$F_4 = \frac{1}{\sqrt{4\pi(\tau + a)}} \exp\left(-\frac{(a+\tau)^2}{2\tau}\right). \tag{S17}$$

Here the exponential factor coincides with the result (22), which was obtained for the same range of parameters by determining of optimal fluctuation.

The solution (S16,S17) shows that the random velocity gradient carries the probability away very quickly from the region $J \lesssim 1$, which corresponds to the initial condition (S15) into the region $J \sim e^\tau \gg 1$. Non-zero values of A do not break the trend. Below we show that the total probability is accumulated in the region $A \ll J$ at no very large times $\tau \ll a_\kappa^2$ (20), that corresponds to $r_\perp \ll r$. Hence, the main influence of the diffusion is the transport in A-direction (increase in $r_\perp$ at fixed r). Thus, only the term $\mathrm{Pe}^{-1} J\partial_A^2 F_4$ should be kept among all diffusion terms in equation (S12). Without changing the physical problem to simplify its mathematical formulation, we assume that the domain of definition is $J > 0$, $A > 0$ instead of the truly domain of definition $J \geq A$. In addition, we neglect A^2 compared to J^2 in the first term of the equation (S12). We arrive to the following equation:

$$\partial_\tau F_4 = \left(\frac{1}{2}\partial_a^2 + \partial_a\right)F_4 + \frac{2}{\mathrm{Pe}} e^{2a}\partial_A^2 F_4 \tag{S18}$$

with the initial condition $F_4|_{t=0} = \exp(-A - J)$. Note that the equation (S15) obtained in the diffusionless limit at $A = 0$ coincides with the equation (S18) at finite Péclet number Pe integrated over A.

Now we solve (S18) using separation of variables. We implement Fourier transform in A-variable,

$$F_4 = \int_0^\infty \mathrm{d}q \, \frac{\cos(qA)}{\sqrt{\pi/2}} \, F_{4q}, \qquad F_{4q} = \int_0^\infty \mathrm{d}A \, \frac{\cos(qA)}{\sqrt{\pi/2}} \, F_4. \tag{S19}$$

The substitution $F_{4q} = \exp(-(k^2+1)\tau/2 - a) f(J,q)$ leads us to modified Bessel equation with imaginary order [3, Eq. (10.45.1)]

$$\left(x^2\partial_x^2 + x\partial_x - x^2 + k^2\right) f = 0, \tag{S20}$$

where $x = 2q\exp(a - a_\kappa)$. In quantum-mechanical terms, the solution we are interested in should decrease in the subbarrier region (at large positive $\ln x$) and should turn into standing wave of unit amplitude in the region of classical motion (at large negative $\ln x$). Thus, the solution is

$$f_{kq}(a) = -\frac{\sqrt{k\,\mathrm{sh}(\pi k)}\,\mathrm{K}_{ik}(x)}{\pi/\sqrt{2}}, \tag{S21}$$

where $a_q = \ln(\sqrt{\mathrm{Pe}}/q) - \gamma_{\mathrm{eu}}$ and the Euler constant $\gamma_{\mathrm{eu}} \approx 0.577$, see [3, Eq. (10.45.7)]. The orthogonality condition for the basis functions (S21) is

$$\int_0^{+\infty} \mathrm{d}k \, f_{kq}(b)\, f_{kq}(a) = \delta(a-b). \tag{S22}$$

Now we can write out the solution,

$$F_4(t,A,J) = \int_0^{+\infty} \mathrm{d}k \int_0^{+\infty} \mathrm{d}q \int_{-\infty}^{+\infty} \mathrm{d}b \,\sqrt{J'}\, e^{-J'-(k^2+1)\tau/2} \frac{k\,\mathrm{sh}(\pi k)\,\mathrm{K}_{2ik}(x)\,\mathrm{K}_{2ik}(x')\,\cos(qA)}{(\pi^2/4)(1+q^2)\sqrt{J}}, \tag{S23}$$

where $x = 2q\sqrt{J/\mathrm{Pe}}$, $x' = 2qe^b/\sqrt{\mathrm{Pe}}$ и $2b = \ln J'$.

For further calculations, note that the approximation

$$\mathrm{K}_{ik}(x) \approx \frac{\sin\left(k\,\mathrm{K}_0(x)\right)}{k}, \qquad \text{at} \quad k \ll 1 \tag{S24}$$

is applicable for $k \ll 1$ at any $x > 0$. We also remind that Macdonald's function of zero order $\mathrm{K}_0(x)$ is positive at positive values of its arguments, and its asymptotics at small and large arguments are

$$\begin{aligned} x \ll 1: &\quad \mathrm{K}_0(x) \approx -\ln\frac{x}{2} - \gamma_{\mathrm{eu}} = a_q - a, \\ x \gg 1: &\quad \mathrm{K}_0(x) \approx \sqrt{\pi/(2x)}\, e^{-x}. \end{aligned} \tag{S25}$$

We use the approximation (S24) to simplify (S23) to a form which is more convenient for further integrations:

$$F_4 = \int \mathrm{d}k \int_0^{+\infty} \mathrm{d}q \int_{-\infty}^{+\infty} \mathrm{d}b \,\sqrt{J'} e^{-J'-(k^2+1)\tau/2} \frac{\sin\left(2k\,\mathrm{K}_0(x)\right)\,\sin\left(2k\,\mathrm{K}_0(x')\right)\cos(qA)}{(\pi/2)\,(1+q^2)\sqrt{J}}, \tag{S26}$$

The integration is saturated at $x' \ll 1$, so we use approximation (S25) and implement integration over J':

$$\mathrm{Im}\left(e^{-ika_q}\int_0^{+\infty} \mathrm{d}J'\, J'^{(ik-1)/2}\, e^{-J'}\right) \approx \sin\left(k\, a_q\right). \tag{S27}$$

As a result, only two integration remain:

$$F_4 = \int \mathrm{d}k \int_0^{+\infty} \mathrm{d}q \, \frac{\sin(k\mathrm{K}_0)\sin(ka_q)\cos(qA)}{2\pi\,(1+q^2)\sqrt{J}}\, e^{-(k^2+1)\tau/2},$$

where $\mathrm{K}_0 = \mathrm{K}_0(x)$. After we integrate over k, we obtain single integral

$$F_4 = \frac{e^{-\tau/2}}{4\sqrt{2\tau J}} \int_0^{+\infty} \frac{\mathrm{d}q\,\cos(qA)}{(\sqrt{\pi}/2)\,(1+q^2)} \exp\left(-\frac{\mathrm{K}_0^2 + a_q^2}{2\tau}\right)\mathrm{sh}\left(\frac{\mathrm{K}_0\, a_q}{\tau}\right). \tag{S29}$$

Now we start to evaluate the integral (S29) in different regions of the variables J and A. We expand $2\,\mathrm{sh}(\cdot) = \exp(\cdot) - \exp(-\cdot)$ and recast the right-hand side of (S29) as

$$F_4 = F_{4+} + F_{4-}. \tag{S30}$$

The part $F_{4-}(t,A,J)$ tends to zero in the limit of zero molecular diffusion, i.e. when $\mathrm{Pe} \to \infty$, and the rest part $F_{4+}(t,A,J)$ cease to depend on Péclet number Pe in the limit.

In the region of the variables t, J, A where F_{4+} is independent of Pe, this part of the probability density function is proportional to the solution (S17) obtained in the diffusionless limit. Let us prove the statement. Under the condition, the arguments of sh and K_0 are large in the region where the integral (S29) is saturated. Then the product exp · sh in (S29) does not depend on q and is equal to $\exp(-a^2/(2\tau))$. Further integration over q is saturated at $q_\star \sim \min(1, A^{-1})$ and reproduces the initial factor $\exp(-A)$ posed in the initial condition. The conditions $\mathrm{K}_0 \gg 1$ means $J \ll \mathrm{Pe}(A^2+1)$; the region is bounded from below by black dashed line in Fig. S1. In terms of variables J and A, the restriction is always weaker than the restriction $\mathrm{K}_0 a_q/\tau \gg 1$. Hence, it is sufficient to claim only the last restriction, which is equivalent to

$$\tau \ll \mathrm{K}_0 a_{q_\star} = \big(a_{q_\star} - a\big) a_{q_\star} \tag{S31}$$

$$\text{or} \qquad a < a_\kappa - \ln q_\star - \frac{\tau}{a_\kappa - \ln q_\star}. \tag{S32}$$

The corresponding area is bounded from below by bold green dashed line in Fig. S1 for moderately large times, $\tau \ll a_\kappa^2$. (Here we remind that the condition $\tau \ll a_\kappa^2$ was already obtained before, see (20). Its meaning is to make it possible to use the optimal fluctuation approximation when estimating the probability density function.) We arrive to the dependence

$$F_{4+} \;=\; \frac{1}{2\sqrt{2\pi\tau}} \exp\left(-\frac{(a+\tau)^2}{2\tau} - A\right), \tag{S33}$$

which expands (S17) for nonzero A. The exponential dependence on A in (S33) corresponds to the δ-functional-like peak dependence on $r_\perp$, established earlier in the main text by searching for the optimal fluctuation, see (22) and (25,27).

Next, we evaluate the second term F_{4-} in (S30) in the region above the black dashed line in Fig. S1 at extra condition $A \gg 1$. The region of q where the integral (S29) is saturated is determined by the factor $\cos(qA)$, and the approximation (S25) should be used for $\mathrm{K}_0 \gg 1$. We use also the change $\cos(qA) \to \exp(iqA)$ and obtain

$$F_{4-} \;=\; -\frac{e^{-\tau/2}}{8\sqrt{2\tau J}} \\ \mathrm{Re}\int\limits_0^{+\infty} \frac{\mathrm{d}q\, \exp(iqA)}{(\sqrt{\pi}/2)\,(1+q^2)} \exp\left(-\frac{(2a_q - a)^2}{2\tau}\right). \tag{S34}$$

On the complex q-plane, we rotate the integration contour in the vicinity of zero point counterclockwise, so the contour becomes vertically oriented, so $q = i|q|$. Under the contour deformation, $a_q \to a_{|q|} - i\pi/2$. Now the integral is evaluated as

$$F_{4-} \;=\; \frac{\ln\big(\mathrm{Pe}(A^2+1)\sqrt{J}\big)}{8\sqrt{2\pi\tau^3}} \\ \frac{1}{A\sqrt{J}} \exp\left(-\frac{1}{2\tau}\ln^2\frac{\mathrm{Pe}(A^2+1)}{\sqrt{J}} - \frac{\tau}{2}\right). \tag{S35}$$

(Note, that if the factor dependent on ς_q is absent in the integrand, then this way of the integral calculation is inapplicable since it produces zero instead of exponentially small $\exp(-A)$.) The factor written in the second line in (S35) coincides with the obtained before in the main text dependence (25) by searching of the optimal fluctuation.

Finally consider the region $J \gg \mathrm{Pe}(A^2+1)$, which corresponds to distances $r \gg l^2/r_\kappa$ and $r_\perp \ll r_\kappa$. The region is placed below the black dashed line in Fig. S1. It should be expected that the correlation function does not depend on the smaller length $r_\perp$ in this case. Starting with the calculations, we note that the inequality $\tau \ll a^2$ is also true in the considered limit (20) of moderately large times, since $a > a_\kappa$. The region of q where the integral is saturated in defined by the inequality $\mathrm{K}_0 \gtrsim \max(1, \tau/a_q)$ (i.e. $F_{4+} \gg F_{4-}$ in the division (S30)), which is equivalent to the condition $q \lesssim l^2/r_\kappa r$ at moderately large times. Thus, we arrive to the estimate (22) instead of (S17), and (22) was obtained by determining of the optimal fluctuation (22).

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