# Supplementary Material to the article "Generalized Bloch theorem and band structure topology"

**1. Dimensionality of state space.**

In the 1D и 2D models considered a state with a fixed $\mathbf{k}$ vector is described by 6 complex variables (3 sublattices, 2 spin projections), i.e. it is a point in the $C^6$ space. After a transition to real variables one obtain $n$-sphere ($S^{11}$) taking into account conservation of a vector norm.

**2. Isotopy and structure of the multisheeted surface.**

Two smooth embeddings $f, g : M \to N$ are isotopic if there exists a smooth homotopy $H : M \times I \to M$ (here $I$ is a unit interval, $t \in [0, 1]$) such that $H_t$ is an embedding at any $t$, $H_0 = f$ and $H_0 = g$.

n the case of 2D dispersion surfaces considered in the work, they are embeddings of a torus into the state space $S^1 \times S^1 \times S^n$. Moreover, the homotopy classes of equivalence for the paths of these embeddings are determined by the homotopy classes of paths of the state space.

The path $P$ (or $Q$) defines monodromy as a cyclic permutation of three sheets. It follows, in particular, that the covering (multi-sheet dispersion surface) is isotopic to surfaces, which are also coverings with the same number of sheets. Thus, the number of covering sheets is a topological invariant for isotopic surfaces.

One can narrow the definition of isotopy to the so-called ambient isotopy. Then the conclusion above becomes even more obvious.

Two smooth embeddings $f, g : M \to N$ are ambient isotopic if there exists a smooth homotopy $H : N \times I \to N$ such that $H_t$ is a homeomorphism at any $t \in I$, $H_0 = \mathrm{id}_N$ and $H_1 \circ f = g$.

**3. Stability of the topological structure under a uniform magnetic field.**

**1D model.**

The dispersion for the 1D model with $120^0$ ordering is shown in Fig.S1a (corresponds to Fig.2). Dispersion curves are also shown in the case of applying an additional homogeneous magnetic field perpendicular to the helicoid plane (Fig.S1b), i.e. a cone total field, as well as in the helicoid plane (Fig.S1c).

**2D model.**

A three-dimensional view of the dispersion surfaces of the 2D model is shown in Fig.S2a. Crossings of sheets of the dispersion surface, which correspond to

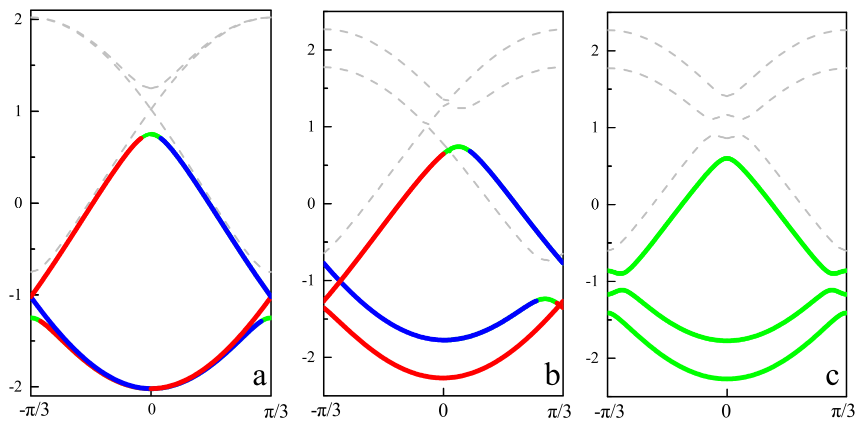


Fig.S1. Dispersion curves in the 1D model at $h_0 = 0.25$: (a) $120^0$ magnetic field only, (b) additional uniform magnetic field along the $z$ axis ($h_z = 0.25$), (c) additional uniform magnetic field along the $x$ axis ($h_x = 0.25$). The spin structure for the lower branch is shown in same colors as in Fig.2.

a topologically nontrivial state, are seen. When an additional magnetic field is applied in the helical plane, the band structure becomes topologically trivial (Fig. S2b).

**2. The Fermi surface of the 2D model.**

The Fermi surface of the 2D model is shown in Fig.S3. It is similar to that obtained earlier in the model of almost free electrons.

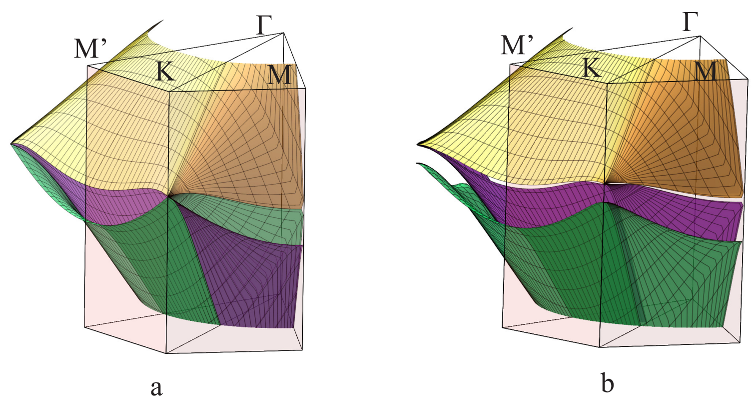


Fig.S2. Dispersion surfaces near the boundary of the hexagonal magnetic Brillouin zone for a two-dimensional model at $h_0 = 1$ (corresponding to the low branches in Fig.3): (a) the $120^0$ magnetic field only, (b) the additional magnetic field along the $x$ axis ($h_x = 0.25$).

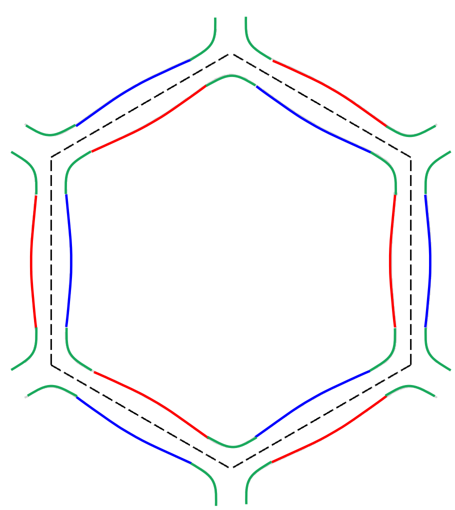

Fig.S3. The spin texture of the Fermi surface corresponding to the Fermi level in Fig. 3 (dashed-dotted line). The colors correspond to the palette in Fig. 2 and Fig. 3.