Supplementary Material to the article "Effect of Sum Blockade in a Polariton Trimer"

1. FLOQUET ENGINEERING OF THE POLARITON TRIMER HAMILTONIAN

Consider the following Hamiltonian for a model describing three tunnel-coupled polariton modes formed in three closely spaced micropillars, interacting with each other due to the overlap of the polariton wave functions of neighboring micropillars. We also introduce modulation of the natural frequency of the micropillars following a cosine law:

$$\begin{aligned}&\hat{H}_0 = \textstyle\sum_{i=1}^{3} \omega \hat{a}_i^\dagger \hat{a}_i + A_1 \cos\left(w_1 t + \varphi_1\right) \hat{a}_1^\dagger \hat{a}_1 \\ &+ A_2 \cos\left(w_2 t + \varphi_2\right) \hat{a}_2^\dagger \hat{a}_2 + A_3 \cos\left(w_3 t + \varphi_3\right) \hat{a}_3^\dagger \hat{a}_3 \\ &+ \textstyle\sum_{i \neq j}^{3} g_{ij} \hat{a}_i^\dagger \hat{a}_j + \text{h.c.} \\ &+ \textstyle\sum_{i=1}^{3} U_{nl} \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i + (\Omega(t) \hat{a}_1^\dagger + \Omega(t)^* \hat{a}_1),\end{aligned} \tag{S1}$$

where $\Delta = (\omega - \omega_d)$.

Transform the Hamiltonian (S1) using the unitary operator

$$\hat{U}(t) = \exp\left\{ \sum_{j=1}^{3} \theta_j \hat{a}_j^\dagger \hat{a}_j \right\}, \tag{S2}$$

where $\theta_j = \frac{A_j}{w_j} \sin\left(w_j t + \varphi_j\right) + \omega_d t$, and using the following transformation rule for the Hamiltonian

$$\hat{H}' = \hat{U} \hat{H}_0 \hat{U}^\dagger + i \frac{d\hat{U}}{dt} \hat{U}^\dagger, \tag{S3}$$

we obtain the following Hamiltonian:

$$\begin{aligned}\hat{H}' = &\sum_{i=1}^{3} \Delta \hat{a}_i^\dagger \hat{a}_i + \sum_{i=1}^{3} U_{nl} \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i \\ &+ g_{21} \hat{a}_2^\dagger \hat{a}_1 \exp(i\theta_2 - i\theta_1) + \text{h.c.} \\ &+ g_{23} \hat{a}_2^\dagger \hat{a}_3 \exp(i\theta_2 - i\theta_3) + \text{h.c.} \\ &+ g_{31} \hat{a}_3^\dagger \hat{a}_1 \exp(i\theta_3 - i\theta_1) + \text{h.c.} \\ &+ \Omega'(t) \hat{a}_1^\dagger + \Omega'(t)^* \hat{a}_1,\end{aligned} \tag{S4}$$

where $\Omega'(t) = \Omega(t) e^{i\omega_d t + i\frac{A_1}{w_1}\sin(w_1 t + \phi_1)}$. Apply the Jacobi-Anger expansion:

$$e^{iz\sin\alpha} = \sum_{n=-\infty}^{\infty} J_n(z) e^{in\alpha}, \tag{S5}$$

to obtain the following Hamiltonian:

$$\begin{aligned}\hat{H}' = &\sum_{i=1}^{3} \Delta \hat{a}_i^\dagger \hat{a}_i + \sum_{i=1}^{3} U_{nl} \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i \\ &+ g_{12} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_n\left(\frac{A_1}{w_1}\right) J_m\left(\frac{A_2}{w_2}\right) \hat{a}_1^\dagger \hat{a}_2 e^{iw_{12}^{nm} t} e^{i\phi_{12}^{nm}} \\ &+ g_{13} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_n\left(\frac{A_1}{w_1}\right) J_m\left(\frac{A_3}{w_3}\right) \hat{a}_1^\dagger \hat{a}_3 e^{iw_{13}^{nm} t} e^{i\phi_{13}^{nm}} \\ &+ g_{23} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_n\left(\frac{A_2}{w_2}\right) J_m\left(\frac{A_3}{w_3}\right) \hat{a}_2^\dagger \hat{a}_3 e^{iw_{23}^{nm} t} e^{i\phi_{23}^{nm}} \\ &+ \Omega'(t) \hat{a}_1^\dagger + \Omega'(t)^* \hat{a}_1,\end{aligned} \tag{S6}$$

where $w_{ij}^{nm} = w_i n - w_j m$, $\phi_{ij}^{nm} = \phi_i n - \phi_j m$. The terms with g_{ij} are stationary if the frequency in the exponent is zero:

$$w_{ij}^{nm} = n w_i - m w_j = 0 \tag{S7}$$

Only such terms contribute to the effective hopping of a polariton between micropillar i and j. With (S7), we obtain the following effective coupling constant:

$$g_{ij}^{\text{eff}} = g_{ij} \sum_{n,m \,\mid\, n w_i = m w_j} J_n\left(\frac{A_i}{w_i}\right) J_m\left(\frac{A_j}{w_j}\right) e^{i(n\phi_i - m\phi_j)}. \tag{S8}$$

Assuming that the external laser pumping is also modulated: $\Omega(t) = F_1 e^{-i\omega_d t} e^{-i\frac{A_1}{w_1}\sin(w_1 t + \phi_1)}$. Then, in the rotating wave approximation, we obtain the following Hamiltonian:

$$\begin{aligned}\hat{H} = &\sum_{i=1}^{3} \Delta \hat{a}_i^\dagger \hat{a}_i + \sum_{i=1}^{3} U_{nl} \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i \\ &+ \left(g_{12}^{\text{eff}} \hat{a}_1^\dagger \hat{a}_2 + g_{13}^{\text{eff}} \hat{a}_1^\dagger \hat{a}_3 + g_{23}^{\text{eff}} \hat{a}_2^\dagger \hat{a}_3 + \text{h.c.} \right) \\ &+ F_1 \left(\hat{a}_1^\dagger + \hat{a}_1 \right).\end{aligned} \tag{S9}$$

The coupling in (S9) is non-reciprocal, but the Hamiltonian remains Hermitian, as $\hat{H} = \hat{H}^\dagger$.

Under the condition $w_i = 2w_j$ (i>j), the resonance condition $n w_i = m w_j$ becomes a condition on the indices $2n = m$. The smallest integer solutions (besides the trivial one) are $(n, m) = (1, 2)$ and $(n, m) = (-1, -2)$.

The effective coupling is the sum over all multiple solutions:

$$g_{ij}^{\text{eff}} = g_{ij} \sum_{k=-\infty}^{\infty} J_k\left(\frac{A_i}{w_i}\right) J_{2k}\left(\frac{A_j}{w_j}\right) e^{ik(\phi_i - 2\phi_j)} \tag{S10}$$

For small arguments of the Bessel functions $\frac{A}{w}$, the value of $J_n(z)$ decreases rapidly with increasing index n. This allows truncation of the infinite series, retaining only the most significant terms corresponding to the smallest indices k, namely: $J_0\left(\frac{A_i}{w_i}\right) J_0\left(\frac{A_j}{w_j}\right)$, $J_1\left(\frac{A_i}{w_i}\right) J_2\left(\frac{A_j}{w_j}\right) e^{i(\phi_i-2\phi_j)}$, and $J_{-1}\left(\frac{A_i}{w_i}\right) J_{-2}\left(\frac{A_j}{w_j}\right) e^{-i(\phi_i-2\phi_j)}$.

Thus, the series can be approximated with high accuracy by the sum of these three terms:

$$\begin{aligned} \frac{g_{ij}^{\text{eff}}}{g_{ij}} &\approx J_0\left(\frac{A_i}{w_i}\right) J_0\left(\frac{A_j}{w_j}\right) \\ &+ J_1\left(\frac{A_i}{w_i}\right) J_2\left(\frac{A_j}{w_j}\right) e^{i\Phi} + J_{-1}\left(\frac{A_i}{w_i}\right) J_{-2}\left(\frac{A_j}{w_j}\right) e^{-i\Phi_{ij}}, \end{aligned} \tag{S11}$$

where $\Phi_{ij} = \phi_i - 2\phi_j$.

Using the identity $J_{-n}(z) = (-1)^n J_n(z)$, we obtain the following expression for the effective coupling parameter:

$$\frac{g_{ij}^{\text{eff}}}{g_{ij}} \approx J_0(\mathscr{A})^2 + 2iJ_1(\mathscr{A})\, J_2(\mathscr{A}) \sin(\Phi_{ij}) = \mathscr{G}_{ij} e^{i\Theta_{ij}}, \tag{S12}$$

where we chose modulation amplitudes A_i such that $\frac{A_i}{w_i} = \mathit{const} \equiv \mathscr{A} \ll 1$, and introduced the notations: $\mathscr{G}_{ij} = \sqrt{J_0(\mathscr{A})^4 + 4J_1(\mathscr{A})^2 J_2(\mathscr{A})^2 \sin^2(\Phi_{ij})}$ and $\Theta_{ij} = \arctan\left(\frac{2J_1(\mathscr{A})J_2(\mathscr{A})\sin(\Phi_{ij})}{J_0(\mathscr{A})^2}\right)$.

2. SOLUTION OF THE SCHRÖDINGER EQUATION

Consider the following Bose-Hubbard model, which describes our model in the particular case of equal detunings. The Hamiltonian of such a model is given by:

$$\begin{aligned} \hat{H} =& \sum_{i=1}^{3} \Delta_i \hat{a}_i^\dagger \hat{a}_i + \sum_{i=1}^{3} U \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i \\ &+ \left(g_{12}\hat{a}_1^\dagger \hat{a}_2 + g_{23}\hat{a}_2^\dagger \hat{a}_3 + g_{31}\hat{a}_3^\dagger \hat{a}_1 + \text{h.c.}\right) \\ &+ \sum_i F_i \left(\hat{a}_i^\dagger + \hat{a}_i\right). \end{aligned} \tag{S13}$$

The state of the system $|\psi\rangle$ can be written as a sum of n-particle states $|\psi^{(n)}\rangle$, each representing a superposition of states $|ijk\rangle$, describing different distributions of n particles across the three modes of the system:

$$|\psi\rangle = \sum_{n \le M} |\psi^{(n)}\rangle = \sum_{n \le M} \sum_{i+j+k=n} c_{ijk} |ijk\rangle, \tag{S14}$$

where c_{ijk} is the probability amplitude for the state $|ijk\rangle$, with $i+j+k = n$, denoting a state with i particles in the first mode, j particles in the second mode, and k particles in the third mode. In the first equality, the n-particle quantum state is introduced; M is the size of the truncated Hilbert space (maximum number of particles in the system). From the Hamiltonian (S9), we can obtain the following system of differential equations for the amplitudes c_{ijk} of the state (S6):

$$\begin{aligned} & i\frac{dc_{ijk}}{dt} = \Big[\tilde{\Delta}_1 i + \tilde{\Delta}_2 j + \tilde{\Delta}_3 k + \\ & U\left(i(i-1) + j(j-1) + k(k-1)\right)\Big] c_{ijk} + \\ & F_1\sqrt{i+1}c_{(i+1)jk} + F_1\sqrt{i}c_{(i-1)jk} + \\ & F_2\sqrt{j+1}c_{i(j+1)k} + F_2\sqrt{j}c_{i(j-1)k} + \\ & F_3\sqrt{k+1}c_{ij(k+1)} + F_3\sqrt{k}c_{ij(k-1)} + \\ & g_{12}\sqrt{i(j+1)}c_{(i-1)(j+1)k} + g_{21}\sqrt{j(i+1)}c_{(i+1)(j-1)k} + \\ & g_{13}\sqrt{i(k+1)}c_{(i-1)j(k+1)} + g_{31}\sqrt{k(i+1)}c_{(i+1)j(k-1)} + \\ & g_{23}\sqrt{j(k+1)}c_{i(j-1)(k+1)} + g_{32}\sqrt{k(j+1)}c_{i(j+1)(k-1)}, \end{aligned} \tag{S15}$$

where $\tilde{\Delta}_j = \Delta_j - i\frac{\gamma_j}{2}$.

The Schrödinger equation for the n-particle state $|\psi^{(n)}(t)\rangle$ can be written as:

$$i\frac{d}{dt}|\psi^{(n)}(t)\rangle \approx H_0|\psi^{(n)}(t)\rangle + H_+|\psi^{(n-1)}(t)\rangle, \tag{S16}$$

where we introduced the particle-number-conserving part of the full non-Hermitian Hamiltonian $\hat{H}_0 = \hat{\mathbf{a}}^\dagger \mathbf{G}_{\text{eff}}^{(1)} \hat{\mathbf{a}} + \sum_i U_i \hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i$ and the part increasing the particle number by one: $\hat{H}_+ = \sum_i F_i \hat{a}_i^\dagger$. Here, $\mathbf{G}_{\text{eff}}^{(1)} = \mathbf{G}^{(1)} - i/2\,\mathrm{diag}(\gamma_1, \gamma_2, \gamma_3)$ is the effective non-Hermitian coupling matrix for the single-particle subspace. Additionally, we neglected the pumping part of the Hamiltonian H_+,which increases the particle number, which is valid in the weak pumping limit due to the leading order of probability amplitudes $\|\mathbf{c}^{(n+1)}\| \ll \|\mathbf{c}^{(n)}\|$.

From (S16) for the amplitude vector of the n-particle state

$$\begin{aligned} \mathbf{c}^{(n)} = (&c_{0,n,0}, c_{0,0,n}, c_{n-1,1,0}, c_{n-1,0,1}, c_{n-2,2,0}, \\ &c_{n-2,1,1}, c_{n-2,0,2}, \dots, c_{0,1,n-1}, c_{0,0,n})^{\mathsf{T}}, \end{aligned} \tag{S17}$$

we can obtain the following equation

$$i\frac{d\mathbf{c}^{(n)}}{dt} = \mathbf{G}^{(n)}\mathbf{c}^{(n)} + \mathbf{f}^{(n)}\mathbf{c}^{(n-1)}, \tag{S18}$$

the amplitude vector $\mathbf{c}^{(n)}$ contains T_{n+1} elements.

For the two-particle state, the matrix $\mathbf{f}$ takes the form:

$$\mathbf{f}^{(2)} = \begin{bmatrix} \sqrt{2}F_1 & 0 & 0 \\ 0 & \sqrt{2}F_2 & 0 \\ 0 & 0 & \sqrt{2}F_3 \\ F_2 & F_1 & 0 \\ F_3 & 0 & F_1 \\ 0 & F_3 & F_2 \end{bmatrix} \tag{S19}$$

while the matrix preserving the particle number $n = 2$ is:

$$\mathbf{G}^{(2)} = \begin{bmatrix} 2\tilde{\Delta}_1 + 2U & 0 & 0 & \sqrt{2}g_{21} & \sqrt{2}g_{31} & 0 \\ 0 & 2\tilde{\Delta}_2 + 2U & 0 & \sqrt{2}g_{12} & 0 & \sqrt{2}g_{32} \\ 0 & 0 & 2\tilde{\Delta}_3 + 2U & 0 & \sqrt{2}g_{13} & \sqrt{2}g_{23} \\ \sqrt{2}g_{12} & \sqrt{2}g_{21} & 0 & \tilde{\Delta}_1 + \tilde{\Delta}_2 & 0 & 0 \\ \sqrt{2}g_{31} & 0 & \sqrt{2}g_{13} & 0 & \tilde{\Delta}_1 + \tilde{\Delta}_3 & 0 \\ 0 & \sqrt{2}g_{32} & \sqrt{2}g_{23} & 0 & 0 & \tilde{\Delta}_2 + \tilde{\Delta}_3 \end{bmatrix} \tag{S20}$$

Using the obtained matrices $\mathbf{G}, \mathbf{f}$, we can find stationary solutions of the equations (S18) for the probability amplitudes $\mathbf{c}^{(n)}$ recursively:

$$\begin{aligned} \mathbf{c}^{(1)} &= -\mathbf{G}_{eff}^{-1}\mathbf{f}^{(1)}, \\ \mathbf{c}^{(2)} &= -\mathbf{G}^{(2)^{-1}}\mathbf{f}^{(2)}\mathbf{c}^{(1)} = \mathbf{G}^{(2)^{-1}}\mathbf{f}^{(2)}\mathbf{G}_{eff}^{-1}\mathbf{f}^{(1)}. \end{aligned} \tag{S21}$$

The normalized wave function takes the form:

$$|\psi_{\text{norm}}\rangle = \frac{\sum_{n \leq M} \sum_{i+j+k=n} c_{ijk}|ijk\rangle}{\sqrt{\sum_{i,j,k} |c_{ijk}|^2}}. \tag{S22}$$

3. STATIONARY SOLUTION OF THE LINDBLAD EQUATION

We write the Lindblad equation for the trimer:

$$\frac{\partial \rho}{\partial t} = \mathcal{L}\rho_{ss} = -i\left[\hat{H}, \rho_{ss}\right] + \sum_{j=1}^{3} \gamma \mathcal{D}_j\left[\rho_{ss}\right], \tag{S23}$$

where $\mathcal{D}_j[\rho] = \hat{a}_j \rho \hat{a}_j^\dagger - \frac{1}{2}\{\hat{a}_j^\dagger \hat{a}_j, \rho\}$, and $\{\cdot, \cdot\}$ denotes the anticommutator.

The equations for the matrix elements of the density matrix $\rho_{\mathbf{nm}}$ are:

$$\begin{aligned} \frac{\partial \rho_{nm}}{\partial t} =& -i\sum_k \left(H_{\mathbf{nk}}\rho_{\mathbf{km}} - \rho_{\mathbf{nk}}H_{\mathbf{km}}\right) \\ &+ \sum_{j=1}^{3} \gamma_j \left[\sqrt{(n_j+1)(m_j+1)}\, \rho_{n_j \to n_j+1}^{m_j \to m_j+1} \right. \\ &\left. - \frac{n_j + m_j}{2}\rho_{\mathbf{nm}}\right], \end{aligned} \tag{S24}$$

where $\rho_{n_j \to n_j \pm 1}^{m_j \to m_j \pm 1}$ denotes $\rho_{n_1,\ldots,n_j\pm1,\ldots}^{m_1,\ldots,m_j\pm1,\ldots}$.

The contributions to the equation (S24) are separated by their physical meaning as follows:

1. Diagonal part of the Hamiltonian:

$$\begin{aligned} -i\Bigg[\sum_{i=1}^{3}\big(\Delta(n_i - m_i) \\ +U(n_i(n_i-1) - m_i(m_i-1))\big)\Bigg]\rho_{\mathbf{nm}} \end{aligned}$$

2. Coupling between modes:

$$\begin{aligned} -i\sum_{i\neq j}\Bigg[g_{ij}\sqrt{n_i(m_j+1)}\rho_{n_i\to n_i-1, m_j\to m_j+1} \\ -g_{ji}\sqrt{m_i(n_j+1)}\rho_{n_j\to n_j+1, m_i\to m_i-1}\Bigg] \end{aligned}$$

3. Laser pumping:

$$\begin{aligned} -iF_1\Big[\sqrt{n_1}\rho_{n_1-1} - \sqrt{m_1}\rho_{m_1-1} \\ +\sqrt{n_1+1}\rho_{n_1+1} - \sqrt{m_1+1}\rho_{m_1+1}\Big] \end{aligned}$$

4. Dissipation:

$$\begin{aligned} +\sum_{j=1}^{3}\gamma_j\Bigg[\sqrt{(n_j+1)(m_j+1)}\rho_{n_j\to n_j+1, m_j\to m_j+1} \\ -\frac{n_j+m_j}{2}\rho_{\mathbf{nm}}\Bigg] \end{aligned}$$

To find the stationary state ρ_{ss} ($t \to \infty$) of the polariton system, it is necessary to solve the Lindblad equation with the condition $\frac{\partial \rho}{\partial t} = 0$. This transforms the dynamic equation into a homogeneous system of linear equations $\mathcal{L}\rho_{\text{ss}} = 0$, where $\mathcal{L}$ is the matrix form of the Liouville superoperator.

In the basis of occupation numbers, the Hamiltonian and dissipation operators couple only states with nearby particle numbers, so the matrix form of $\mathcal{L}$ is highly sparse. This property is crucial for computational efficiency, as the matrix dimension, equal to $(M+1)^{2k}$ (where $k = 3$ is the number of modes, and M is the maximum number of particles per mode used for truncation of the Fock space), can be very large. To solve this system, a direct method based on LU decomposition for sparse matrices was used. This approach is implemented in the `steadystate` function of the QuTiP (Quantum Toolbox in Python) library, which efficiently finds the desired non-trivial zero eigenvector and automatically normalizes it to satisfy $\text{Tr}(\rho_{\text{ss}}) = 1$.