Supplementary Material to the article
"A non-magnetic mechanism of backscattering in helical edge states"

I. SCATTERING WAVES AND FIELD OPERATORS IN THE NON-INTERACTING CASE

In the non-interacting case, the scattering waves describing electrons moving from two leads read

$$|\psi_k^{\rm a}\rangle = \begin{cases} e^{ikx}\,|\uparrow\rangle\,,\; x<0, \\ \left[-fe^{iky}\,|\downarrow\rangle + re^{ik(L-y)}\,|\uparrow\rangle\right] D(k)\,,\; 0<y<L, \\ e^{ikx}e^{i\theta_k}\,|\uparrow\rangle\,,\; x>0, \end{cases}$$
$$|\psi_k^{\rm b}\rangle = \begin{cases} e^{-ikx}e^{i\theta_k}\,|\downarrow\rangle\,,\; x<0, \\ -\left[re^{iky}\,|\downarrow\rangle + fe^{ik(L-y)}\,|\uparrow\rangle\right] D(k)\,,\; 0<y<L, \\ e^{-ikx}\,|\downarrow\rangle\,,\; x>0. \end{cases} \tag{S1}$$

Here,

$$D(k) = \frac{1}{1-te^{ikL}}\,, \qquad e^{i\theta_k} = (t-e^{ikL})D(k)\,, \tag{S2}$$

and $|\uparrow\rangle$ and $|\downarrow\rangle$ are orthogonal spinors describing left- and right-moving edge states. Since we use basis of scattering waves, we can assume below that $k>0$.

Annihilation field-operator $\hat\Psi$ in a certain coordinate point is given by

$$\hat\Psi(s) = \int \frac{dk}{2\pi}\left(\hat c_k^{\rm a}|\psi_k^{\rm a}\rangle + \hat c_k^{\rm b}|\psi_k^{\rm b}\rangle\right)\,, \tag{S3}$$

where $s=x$ inside the edge of the TI, $s=y$ in the edge of the island, and $\hat c_k^{\rm a(b)}$ are the electron annihilation operators in the reservoirs $\mu_{\rm a(b)}$.

Inside the puddle edge, for $s=y$, the field operator can be rewritten as follows

$$\hat\Psi(y) = \hat\Psi^{\rm R}(y)|\downarrow\rangle + \hat\Psi^{\rm L}(y)|\uparrow\rangle, \tag{S4}$$

where

$$\begin{aligned}\hat\Psi^{\rm R}(y) &= \int\frac{dk}{2\pi}e^{iky}\big(-f\hat c_k^a - r\hat c_k^b\big)D(k) \\ &= \int\frac{dk}{2\pi}e^{iky}D(k)\sqrt{r^2+f^2}\,c_k^{\rm R},\end{aligned} \tag{S5}$$

is the "right" field operator,

$$\begin{aligned}\hat\Psi^{\rm L}(y) &= \int\frac{dk}{2\pi}e^{ik(L-y)}\big(r\hat c_k^a - f\hat c_k^b\big)D(k) \\ &= \int\frac{dk}{2\pi}e^{ik(L-y)}D(k)\sqrt{r^2+f^2}\,c_k^{\rm L}\end{aligned} \tag{S6}$$

is the "left" field operator, and we introduced chiral annihilation operators

$$\begin{aligned} c_k^{\rm R} &= \frac{-fc_k^{\rm a} - rc_k^{\rm b}}{\sqrt{r^2+f^2}}, \\ c_k^{\rm L} &= \frac{rc_k^{\rm a} - fc_k^{\rm b}}{\sqrt{r^2+f^2}}, \end{aligned} \tag{S7}$$

which obey standard commutation rules

$$\{c_k^{\rm R\dagger}, c_{k'}^{\rm R}\} = \{c_k^{\rm L\dagger}, c_{k'}^{\rm L}\} = 2\pi\delta(k-k'). \tag{S8}$$

Using property

$$\int\frac{dk}{2\pi}e^{ik(y-y')}|D(k)|^2(r^2+f^2) = \delta(y-y'), \tag{S9}$$

which is valid for $0<y<L$ and $0<y'<L$, we find that operators $\hat\Psi^{\rm R,L}$ also have standard commutation rules

$$\{\hat\Psi^{\rm R\dagger}(y), \hat\Psi^{\rm R}(y')\} = \{\hat\Psi^{\rm L\dagger}(y), \hat\Psi^{\rm L}(y')\} = \delta(y-y'). \tag{S10}$$

The densities of the right- and left-moving fermions are expressed in terms of these operators as follows:

$$\hat n_{\rm R} = \hat\Psi^{\rm R\dagger}\hat\Psi^{\rm R}, \quad \hat n_{\rm L} = \hat\Psi^{\rm L\dagger}\hat\Psi^{\rm L}. \tag{S11}$$

Commuting operators $\hat\Psi^{\rm R,L}$ with the total Hamiltonian $H_0 + H_{\rm int}$ and using above equations, we arrive at equations for these operators in the Heisenberg representation:

$$\begin{aligned} i\hbar\frac{\partial\hat\Psi_{\rm R}}{\partial t} &= v_{\rm F}\hat p\,\hat\Psi_{\rm R} + 2\pi\hbar v_{\rm F} g\,\hat n_{\rm L}\hat\Psi_{\rm R}, \\ i\hbar\frac{\partial\hat\Psi_{\rm L}}{\partial t} &= -v_{\rm F}\hat p\,\hat\Psi_{\rm L} + 2\pi\hbar v_{\rm F} g\,\hat n_{\rm R}\hat\Psi_{\rm L}. \end{aligned} \tag{S12}$$

Replacing now operators $\hat n_{\rm R,L}$ with the corresponding quasiclassical densities, we find that Eqs. (S12) coincide with equations for particle moving in the time-dependent matrix potential acting in the island edge, for $0<y<L$:

$$\hat U(y,t) = \begin{pmatrix} U_{\rm L}(y-v_{\rm F}t) & 0 \\ 0 & U_{\rm R}(y+v_{\rm F}t)\end{pmatrix}. \tag{S13}$$

where

$$U_{\rm L} = 2\pi g v_{\rm F} n_{\rm R}(y-v_{\rm F}t)\,, \quad U_{\rm R} = 2\pi g v_{\rm F} n_{\rm L}(y+v_{\rm F}t), \tag{S14}$$

are periodic functions of y and, consequently, t, with the periods L and $L/v_{\rm F}$, respectively. Expanding these functions in the Fourier series, we get

$$\hat U(y,t) = \sum_n \begin{pmatrix} U_{\rm L,n}e^{iq_n(y-v_{\rm F}t)} & 0 \\ 0 & U_{\rm R,n}e^{iq_n(y+v_{\rm F}t)}\end{pmatrix}, \tag{S15}$$

where $q_n = 2\pi n/L$. The term with $n=0$ corresponds to time-independent homogeneous potential acting at $0<y<L$ given by Eq. (12) of the main text.

II. ZERO-MODE DISTRIBUTION FUNCTION

Let us now derive Eqs. (23) and (24) of the main text. Since the total energy (24) is the sum of the "right" and "left" contributions, it is sufficient to derive corresponding expressions for the R-movers. To this end, we write

$$\begin{aligned}\hat{N}_{\mathrm{R}} &= \int_0^L dy\ \hat{n}_{\mathrm{R}} \\ &= \sum_{k,k'} \int_0^L dy D(k) D^*(k') e^{i(k-k')y}(1-t^2) c_{k'}^{\mathrm{R}\dagger} c_k^{\mathrm{R}},\end{aligned} \tag{S16}$$

where $\sum_{k,k'} = \int dk dk'/(2\pi)^2$. The distribution function for N_{R} reads

$$f(N_{\mathrm{R}}) = \left\langle \int \frac{d\varphi}{2\pi} \exp\left[i\varphi \left(N_{\mathrm{R}} - \sum_{k,k'} A_{kk'} c_{k'}^{\mathrm{R}\dagger} c_k^{\mathrm{R}} \right)\right]\right\rangle_{\hat{\rho}}, \tag{S17}$$

where matrix elements of the single-particle operator $\hat{A}$ look

$$A_{kk'} = \int_0^L dy e^{i(k-k')y}(1-t^2) D(k) D^*(k'), \tag{S18}$$

and averaging is taken over the equilibrium density matrix

$$\begin{aligned}\langle \cdots \rangle_{\hat{\rho}} &= \frac{\mathrm{Tr}(\cdots \hat{\rho})}{\mathrm{Tr}(\hat{\rho})}\,, \\ \hat{\rho} &= \exp[-(\hat{H}_0 - \mu)/T]\,.\end{aligned} \tag{S19}$$

Here μ is the chemical potential in the leads. By using trace formula known in the full counting statistics theory (see [1] and Eq. (8) in [2]), we get

$$\left\langle \exp\left[-i\varphi \sum_{k,k'} A_{kk'} c_{k'}^{\mathrm{R}\dagger} c_k^{\mathrm{R}} \right]\right\rangle_{\hat{\rho}} = \det\left[1 - \hat{f} + \hat{f}\ e^{-i\varphi\hat{A}}\right]\,. \tag{S20}$$

Here matrix elements of the operator $\hat{f}$ read

$$f_{kk'} = f_{\mathrm{F}}(k) 2\pi\delta(k-k'), \tag{S21}$$

where $f_{\mathrm{F}}(k) = f_{\mathrm{F}}(E_k)$ is the Fermi distribution function. Using property Eq. (S9), one can check that all eigenvalues of operator $\hat{A}$ equal to unity, so that Eq. (S20) is periodic function of φ with the period 2π, and, consequently, $f(N_{\mathrm{R}})$ is non-zero for integer values of N_{R} as it should be. In order to calculate $f(N_{\mathrm{R}})$ for $T \gg \Delta$, we write $\det[\cdots] = \exp[\ln(\det[\cdots])]$ and expand $\ln(\det[\cdots])$ over φ up to the terms of the second order

$$\begin{aligned}&\ln\left[\det\left(1 - \hat{f} + \hat{f}\ e^{-i\varphi\hat{A}}\right)\right] \\ &\approx \mathrm{Tr}\left(-i\varphi\hat{f}\hat{A} - \frac{\varphi^2}{2}\left[\hat{f}\hat{A}^2 - (\hat{f}\hat{A})^2\right]\right) = -i\varphi\sum_k f_{\mathrm{F}}(k) A_{kk} \\ &- \frac{\varphi^2}{2}\sum_{kk'} f_{\mathrm{F}}(k)[1 - f_{\mathrm{F}}(k')] A_{kk'} A_{k'k} \approx -i\varphi N_0 \\ &- \frac{\varphi^2}{2}\sum_n f_{\mathrm{F}}(k_n)[1 - f_{\mathrm{F}}(k_n)] \approx -i\varphi N_0 - \frac{T}{\Delta}\frac{\varphi^2}{2}.\end{aligned} \tag{S22}$$

Here, $N_0 = L\sum_k f_{\mathrm{F}}(k)$, and $k_n = 2\pi n/L$. While obtaining two bottom lines in Eq. (S22), we took into account that $f_{\mathrm{F}}(k)$ is a smooth function that does not change essentially on the scale $2\pi/L$, so that one can replace $|D(k)|^2 \to \langle |D(k)|^2\rangle_k$, where averaging over k is taken over interval $2\pi/L$. Substituting Eq. (S22) into Eq. (S17), integrating over φ, and performing analogous calculations for the L-mover, we arrive at Eq. (23) of the main text with energy $\epsilon_{N_{\mathrm{R}}N_{\mathrm{L}}}$ given by Eq. (24).

III. SCATTERING WAVES FOR THE HAMILTONIAN $\hat{H}_0 + \hat{U}_0$.

The Hamiltonian $\hat{H}_0 + \hat{U}_0$ with boundary conditions imposed by scattering matrix Eq. (4) describes a single-particle time-independent problem. Solving corresponding Schrodinger equation, we find scattering states

$$\begin{aligned}|\psi_k^{\mathrm{a}}\rangle &= \begin{cases} e^{ikx}\,|\uparrow\rangle + r_{\mathrm{bs}} e^{-ikx}\,|\downarrow\rangle\,,\ x<0, \\ -fD(k_{\mathrm{R}}) e^{ik_{\mathrm{R}}y}\,|\downarrow\rangle + rD(k_{\mathrm{L}}) e^{ik_{\mathrm{L}}(L-y)}\,|\uparrow\rangle\,,\ 0<y<L, \\ Z_{\mathrm{a}} e^{ikx}\,|\uparrow\rangle\,,\ x>0, \end{cases} \\ |\psi_k^{\mathrm{b}}\rangle &= \begin{cases} Z_{\mathrm{b}} e^{-ikx}\,|\downarrow\rangle\,,\ x<0, \\ -rD(k_{\mathrm{R}}) e^{ik_{\mathrm{R}}y}\,|\downarrow\rangle - fD(k_{\mathrm{L}}) e^{ik_{\mathrm{L}}(L-y)}\,|\uparrow\rangle\,,\ 0<y<L, \\ e^{-ikx}\,|\downarrow\rangle + r_{\mathrm{bs}} e^{ikx}\,|\uparrow\rangle\,,\ x>0. \end{cases}\end{aligned} \tag{S23}$$

Here,

$$\begin{aligned}k_{\mathrm{R}} &= k - U_{\mathrm{L},0}/v_{\mathrm{F}}\,, \\ k_{\mathrm{L}} &= k - U_{\mathrm{R},0}/v_{\mathrm{F}}\,, \\ U_{\mathrm{L},0} &= (2\pi g v_{\mathrm{F}}/L) N_{\mathrm{L}}\,, \\ U_{\mathrm{R},0} &= (2\pi g v_{\mathrm{F}}/L) N_{\mathrm{R}}, \\ r_{\mathrm{bs}} &= fr\left[e^{ik_{\mathrm{L}}L} D(k_{\mathrm{L}}) - e^{ik_{\mathrm{R}}L} D(k_{\mathrm{R}})\right]\,, \\ Z_{\mathrm{a}} &= t - r^2 e^{ik_{\mathrm{L}}L} D(k_{\mathrm{L}}) - f^2 e^{ik_{\mathrm{R}}L} D(k_{\mathrm{R}})\,, \\ Z_{\mathrm{b}} &= t - r^2 e^{ik_{\mathrm{R}}L} D(k_{\mathrm{R}}) - f^2 e^{ik_{\mathrm{L}}L} D(k_{\mathrm{L}}),\end{aligned} \tag{S24}$$

and $D(k)$ is given by Eq. (S2)

IV. EFFECT OF THE DYNAMICAL PART OF THE INTERACTION

Let us now consider effect of the dynamical part of the potential Eq. (11)

$$\hat{U}'(y,t) = \sum_{n\neq 0}\begin{pmatrix} U_{\mathrm{L},n} e^{iq_n(y+v_{\mathrm{F}}t)} & 0 \\ 0 & U_{\mathrm{R},n} e^{iq_n(y-v_{\mathrm{F}}t)}\end{pmatrix}. \tag{S25}$$

We expand wave function over non-interacting scattering states, Eq. (S1):

$$\Psi = \sum_k \left(c_k^{\rm a} |\psi_k^{\rm a}\rangle + c_k^{\rm b} |\psi_k^{\rm b}\rangle \right) e^{-iEt}\,, \tag{S26}$$

where coefficients $c_k^{\rm a,b}$ obey

$$i\dot{c}_k^{\alpha} = \sum_{k',\beta} V_{kk'}^{\alpha\beta}(t) c_{k'}^{\beta} e^{i(E_k - E_{k'})t}. \tag{S27}$$

Here $V_{kk'}^{\alpha\beta}(t)$ is time-dependent matrix element of the potential U' corresponding to transition from k', β to k, α. Direct calculation yields the following expressions

$$\begin{aligned} V_{k,k'}^{\rm a,b} e^{i(E_k - E_{k'})t} &= fr \sum_{n\neq 0} \frac{e^{i(k'-k)L} - 1}{i(k' - k + q_n)} e^{iv_{\rm F}t(q_n + k - k')} (U_{{\rm R},n} - U_{{\rm L},-n}) D(k') D^*(k)\,, \\ V_{k,k'}^{\rm b,b} e^{i(E_k - E_{k'})t} &= \sum_{n\neq 0} \frac{e^{i(k'-k)L} - 1}{i(k' - k + q_n)} e^{iv_{\rm F}t(q_n + k - k')} (f^2 U_{{\rm L},-n} + r^2 U_{{\rm R},n}) D(k') D^*(k)\,, \\ V_{k,k'}^{\rm a,a} e^{i(E_k - E_{k'})t} &= V_{k,k'}^{\rm b,b} e^{i(E_k - E_{k'})t}\Big|_{r\leftrightarrow f} \end{aligned} \tag{S28}$$

Let us find probabilities of transition per unit time from k' to k caused by $n-$th harmonics of the dynamical potential, $W_{k,k'}^{\rm a,b}(n)$ and $W_{k,k'}^{\rm b,b}(n)$. Integrating above matrix elements over time from zero to t and squaring the resulting expressions, we obtain the standard golden-rule delta-function $\delta[v_{\rm F}(q_n + k - k')]$. Hence, $k' = k + q_n$. Then, factor standing in front of delta-function turns to zero:

$$\left| \frac{e^{i(k'-k)L} - 1}{i(k' - k + q_n)} \right|^2 \to 0, \quad \text{for} \quad k' = k + q_n. \tag{S29}$$

Hence,

$$W_{k,k'}^{\rm a,b}(n) = W_{k,k'}^{\rm b,b}(n) = 0. \tag{S30}$$

This result justifies the neglect of the dynamical part of the Hamiltonian.

[1] L. S. Levitov, H. Lee, and G. B. Lesovik, J. Math. Phys. **37**, 4845 (1996).

[2] I. Klich, in *Quantum Noise in Mesoscopic Physics*, NATO Science Series, Vol. 97, edited by Y. Nazarov (Springer Science & Business Media, Dordrecht, 2003) pp. 397–402.