Supplementary Material to the article “Phonon mechanism of the anomalous Hall effect in a $p_x + ip_y$ superconductor”

1. Quantum kinetic equations have the form:

$$\hat{G}_0^{-1}\hat{G}^{\lessgtr} - \hat{G}^{\lessgtr}\hat{G}_0^{\dagger -1} = \hat{\Sigma}^{\lessgtr} \otimes \hat{G}^{A} + \hat{\Sigma}^{R} \otimes \hat{G}^{\lessgtr} - \hat{G}^{\lessgtr} \otimes \hat{\Sigma}^{A} - \hat{G}^{R} \otimes \hat{\Sigma}^{\lessgtr}, \tag{S1a}$$

$$\hat{G}_0^{-1}\hat{G}^{R,A} = \hat{\Sigma}^{R,A} \otimes \hat{G}^{R,A}, \tag{S1b}$$

where

$$\hat{G}_0^{-1}(x) = \begin{pmatrix} i\partial_t - \hat{h}(x) & 0 \\ 0 & i\partial_t + \hat{h}^*(x) \end{pmatrix}, \tag{S2}$$

$$\hat{G}(x,x') = \begin{pmatrix} G(x,x') & F(x,x') \\ \tilde{F}(x,x') & \tilde{G}(x,x') \end{pmatrix} = -i \begin{pmatrix} \langle T_c \hat{\psi}^{\dagger}(x)\hat{\psi}(x')\rangle & \langle T_c \hat{\psi}(x)\hat{\psi}(x')\rangle \\ \langle T_c \hat{\psi}^{\dagger}(x)\hat{\psi}^{\dagger}(x')\rangle & \langle T_c \hat{\psi}(x)\hat{\psi}^{\dagger}(x')\rangle \end{pmatrix} \tag{S3}$$

– Green’s function in Nambu space, $\hat{\Sigma}$ – self-energy, T_c – time-ordering operator at Keldysh contour,

$$\hat{h}(x) = \frac{(-i\nabla_{\mathbf{r}} + \mathbf{p}_s(x))^2}{2m} + \Phi(x), \tag{S4}$$

and $\hat{A} \otimes \hat{B} = \int dx_2 \hat{A}(x_1,x_2)\hat{B}(x_2,x_3)$. In what follows, we transform kinetic equations to Wigner coordinates

$$x = \frac{x_1 + x_2}{2}, \quad x' = x_1 - x_2, \tag{S5}$$

and suppose that relative variables, x', vary on microscopic scale while the x are macroscopic and slow variables. Correspondingly, we apply the gradient expansion to convolutions:

$$\int d(x'_{13})\, e^{-i\mathbf{p}\mathbf{r}'_{13} + i\omega t'_{13}} \int dx_2 \hat{A}(x_1,x_2)\hat{B}(x_2,x_3) = \hat{A}(\mathbf{p},\omega;x_{13})\mathcal{G}(\mathbf{p},\omega;x_{13})\hat{B}(\mathbf{p},\omega;x_{13}), \tag{S6}$$

and limit expansion by lowest nonvanishing order. In (S6) the gradient operator $\mathcal{G}$ reads

$$\mathcal{G}(\mathbf{p},\omega;x_{13}) = \exp\left\{\frac{1}{2i}\left[\partial_t^A\partial_\omega^B - \partial_\omega^A\partial_t^B - \partial_{\mathbf{r}}^A\partial_{\mathbf{p}}^B + \partial_{\mathbf{p}}^A\partial_{\mathbf{r}}^B\right]\right\}. \tag{S7}$$

On the left side of the equation (S1a) we have deal with the operator $\hat{g}_0^{-1} - \hat{g}_0^{-1*}$, where $\hat{g}_0^{-1}(x_1) = i\partial_{t_1} - \hat{h}(x_1)$. Within quasiclassical approximation, kinetic operator $\hat{g}_0^{-1} - \hat{g}_0^{-1*}$ take the form

$$\int dx'\, e^{-i\mathbf{p}\mathbf{r}+i\omega t'}\left(\hat{g}_0^{-1}(x_1) - \hat{g}_0^{-1*}(x_2)\right) = i\frac{\partial}{\partial t} + i\tilde{\mathbf{v}}_{\mathbf{p}}\frac{\partial}{\partial \mathbf{r}} - i\left[\frac{\partial\Phi(x)}{\partial\mathbf{r}} + \frac{m}{2}\frac{\partial\tilde{\mathbf{v}}_{\mathbf{p}}^2(x)}{\partial\mathbf{r}}\right]\frac{\partial}{\partial\mathbf{p}} + i\left[\frac{\partial\Phi(x)}{\partial t} + \frac{m}{2}\frac{\partial\tilde{\mathbf{v}}_{\mathbf{p}}^2(x)}{\partial t}\right]\frac{\partial}{\partial\omega}, \tag{S8}$$

where $\tilde{\mathbf{v}}_{\mathbf{p}}(x) = [\mathbf{p} + \mathbf{p}_s(x)]/m$. To construct the Boltzmann theory we will neglect the dependence of distribution function on the relative time and corresponding frequency. Thus we derive truncated version of kinetic operator (S8), which will act on electron Green’s functions:

$$\hat{B} = i\frac{\partial}{\partial t} + i\tilde{\mathbf{v}}_{\mathbf{p}}\frac{\partial}{\partial\mathbf{r}} - i\left[\frac{\partial\Phi(x)}{\partial\mathbf{r}} + \frac{m}{2}\frac{\partial\tilde{\mathbf{v}}_{\mathbf{p}}^2(x)}{\partial\mathbf{r}}\right]\frac{\partial}{\partial\mathbf{p}} \tag{S9}$$

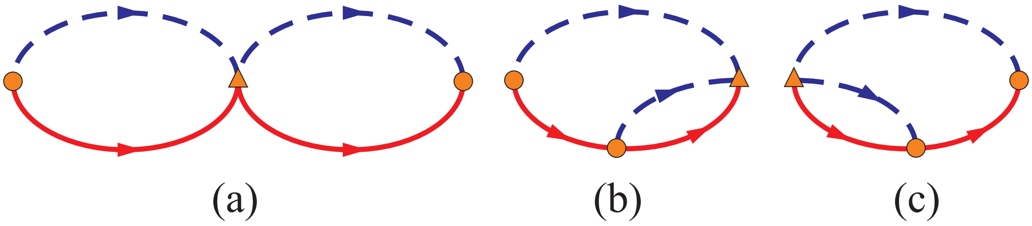

Fig.S1. Self energy $\hat{\Sigma}_{ph}$ diagrams. Red solid and blue dashed lines correspond to electron, $\hat{G}$, and phonon, D, Green's functions correspondingly. Circle and triangle vertexes describe the $\hat{V}_1$ and $\hat{V}_2$ electron-phonon interactions.

2. Self energy is constructed by two terms

$$\hat{\Sigma}(\mathbf{p},\omega;\mathbf{r},t)=\hat{\Delta}_0(\mathbf{p})+\hat{\Sigma}_{ph}(\mathbf{p},\omega;\mathbf{r},t), \tag{S10}$$

where the order parameter $\hat{\Delta}_0$ is a consequence of p_x+ip_y electron-electron pairing and has the form

$$\hat{\Delta}_0(\mathbf{p})=\frac{\Delta_0}{p_F}(\tau_x p_x-\tau_y p_y)=\Delta\begin{pmatrix}0 & e^{i\varphi_{\mathbf{p}}}\\ e^{-i\varphi_{\mathbf{p}}} & 0\end{pmatrix}=\begin{pmatrix}0 & \Delta_{0,\mathbf{p}}\\ \Delta^*_{0,\mathbf{p}} & 0\end{pmatrix} \tag{S11}$$

where $\tau_{x,y,z}$ are the Pauli's matrices. The second term,

$$\hat{\Sigma}_{ph}(\mathbf{p},\omega;\mathbf{r},t)=\begin{pmatrix}\Sigma_{ph} & \Delta_{ph}\\ \tilde{\Delta}_{ph} & \tilde{\Sigma}_{ph}\end{pmatrix}, \tag{S12}$$

describes the electron-phonon collisions and its third-order diagrams are presented in Fig.S1.

3. Bare electron Green's functions are the solution of the equation (S1b) when $\hat{\Sigma}_{ph}$ is omitted:

$$\hat{G}_0^R=\frac{\tau_0(\omega+i\delta)+\tau_z\tilde{\xi}_{\mathbf{p}}+\hat{\Delta}_0(\mathbf{p})}{(\omega+i\delta-\tilde{\varepsilon}_{\mathbf{p}})(\omega+i\delta+\tilde{\varepsilon}_{-\mathbf{p}})}. \tag{S13}$$

Now we can write the lesser and greater Green functions using relations

$$\begin{aligned}\hat{G}^< &= -n_\omega(\hat{G}_0^R-\hat{G}_0^{R\dagger}),\\ \hat{G}^> &= -(n_\omega-1)(\hat{G}_0^R-\hat{G}_0^{R\dagger}),\end{aligned} \tag{S14}$$

where n is Fermi distribution function. The necessary lesser functions are

$$\begin{aligned}G^< &= 2\pi i\big[n_{\mathbf{p}}u_{\mathbf{p}}^2\delta(\omega-\tilde{\varepsilon}_{\mathbf{p}})+(1-n_{-\mathbf{p}})v_{\mathbf{p}}^2\delta(\omega+\tilde{\varepsilon}_{-\mathbf{p}})\big],\\ F^< &= 2\pi i(\Delta/2\varepsilon_{\mathbf{p}})\big[n_{\mathbf{p}}\delta(\omega-\tilde{\varepsilon}_{\mathbf{p}})-(1-n_{-\mathbf{p}})\delta(\omega+\tilde{\varepsilon}_{-\mathbf{p}})\big],\\ \tilde{F}^< &= 2\pi i(\Delta^*/2\varepsilon_{\mathbf{p}})\big[n_{\mathbf{p}}\delta(\omega-\tilde{\varepsilon}_{\mathbf{p}})-(1-n_{-\mathbf{p}})\delta(\omega+\tilde{\varepsilon}_{-\mathbf{p}})\big],\\ \tilde{G}^< &= 2\pi i\big[n_{\mathbf{p}}v_{\mathbf{p}}^2\delta(\omega-\tilde{\varepsilon}_{\mathbf{p}})+(1-n_{-\mathbf{p}})u_{\mathbf{p}}^2\delta(\omega+\tilde{\varepsilon}_{-\mathbf{p}})\big],\end{aligned} \tag{S15}$$

where the Bogolubov's amplitudes are

$$u_{\mathbf{p}}^2=\frac{1}{2}\left(1+\frac{\tilde{\xi}_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\right),\quad v_{\mathbf{p}}^2=\frac{1}{2}\left(1-\frac{\tilde{\xi}_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\right). \tag{S16}$$

The greater electron Green's functions have similar form:

$$\begin{aligned}
G^{>} &= -2\pi i\big[(1-n_{\mathbf{p}})u_{\mathbf{p}}^2\delta(\omega-\tilde{\varepsilon}_{\mathbf{p}}) + n_{-\mathbf{p}}v_{\mathbf{p}}^2\delta(\omega+\tilde{\varepsilon}_{-\mathbf{p}})\big],\\
F^{>} &= -2\pi i(\Delta/2\varepsilon_{\mathbf{p}})\big[(1-n_{\mathbf{p}})\delta(\omega-\tilde{\varepsilon}_{\mathbf{p}}) - n_{-\mathbf{p}}\delta(\omega+\tilde{\varepsilon}_{-\mathbf{p}})\big],\\
\tilde{F}^{>} &= -2\pi i(\Delta^*/2\varepsilon_{\mathbf{p}})\big[(1-n_{\mathbf{p}})\delta(\omega-\tilde{\varepsilon}_{\mathbf{p}}) - n_{-\mathbf{p}}\delta(\omega+\tilde{\varepsilon}_{-\mathbf{p}})\big],\\
\tilde{G}^{>} &= -2\pi i\big[(1-n_{\mathbf{p}})v_{\mathbf{p}}^2\delta(\omega-\tilde{\varepsilon}_{\mathbf{p}}) + n_{-\mathbf{p}}u_{\mathbf{p}}^2\delta(\omega+\tilde{\varepsilon}_{-\mathbf{p}})\big].
\end{aligned} \tag{S17}$$

For the self-energy to be found, the phonon Green functions will be necessary:

$$\begin{aligned}
D^R(\mathbf{p},\omega) &= \frac{\omega_{\mathbf{p}}}{\omega-\omega_{\mathbf{p}}+i\delta} - \frac{\omega_{\mathbf{p}}}{\omega+\omega_{\mathbf{p}}+i\delta},\\
D^{<}(\mathbf{p},\omega) &= -2\pi i\omega_{\mathbf{p}}\left[N_{\mathbf{p}}\delta(\omega-\omega_{\mathbf{p}}) + (1+N_{-\mathbf{p}})\delta(\omega+\omega_{\mathbf{p}})\right],\\
D^{>}(\mathbf{p},\omega) &= -2\pi i\omega_{\mathbf{p}}\left[(1+N_{\mathbf{p}})\delta(\omega-\omega_{\mathbf{p}}) + N_{-\mathbf{p}}\delta(\omega+\omega_{\mathbf{p}})\right],
\end{aligned} \tag{S18}$$

where $N_{\mathbf{p}}$ – phonon distribution function.

4. Boltzmann-equation. Let us introduce the operator

$$\hat{b} = \frac{1}{2\pi i}\int\limits_0^\infty d\left(\omega - \frac{\mathbf{p}\mathbf{p}_s}{m}\right). \tag{S19}$$

Then it is easy to check that

$$\begin{aligned}
n_{\mathbf{p}} &= \hat{b}\left[G^{<}(\mathbf{p},\omega) - G^{>}(-\mathbf{p},-\omega)\right],\\
n_{\mathbf{p}}\frac{\tilde{\xi}_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}} &= \hat{b}\left[G^{<}(\mathbf{p},\omega) + G^{>}(-\mathbf{p},-\omega)\right].
\end{aligned} \tag{S20}$$

Fortunally, operators $\hat{b}$ and $\hat{B}$ obey commutation relation $\hat{b}\hat{B} = \hat{B}\hat{b}$. So we act on the $\{11\}$ component of the kinetic equation (S1a) by $\hat{b}$ and derive Boltzmann-equation using identities (S20)

$$\partial_t n_{\mathbf{p}} + \partial_{\mathbf{p}}\tilde{\varepsilon}_{\mathbf{p}}\partial_{\mathbf{r}}n_{\mathbf{p}} - \partial_{\mathbf{r}}\tilde{\varepsilon}_{\mathbf{p}}\partial_{\mathbf{p}}n_{\mathbf{p}} = I_{st}, \tag{S21}$$

where collision integral reads

$$I_{st} = -i\hat{b}\Big\{[\Sigma(\mathbf{p},\omega)G(\mathbf{p},\omega)]^{<} - [G\Sigma]^{<} + [\Delta\tilde{F}]^{<} - [F\tilde{\Delta}]^{<} - [\Sigma(-\mathbf{p},-\omega)G(-\mathbf{p},-\omega)]^{>} + [G\Sigma]^{>} - [\Delta\tilde{F}]^{>} + [F\tilde{\Delta}]^{>}\Big\}. \tag{S22}$$

By some calculation we get

$$\begin{aligned}
I_{st} = -in_{\mathbf{p}}&\left[u_{\mathbf{p}}^2\Sigma^{>}(1) + v_{\mathbf{p}}^2\tilde{\Sigma}^{>}(1) + \frac{\Delta_{0,\mathbf{p}}^*}{2\varepsilon_{\mathbf{p}}}\Delta^{>}(1) + \frac{\Delta_{0,\mathbf{p}}}{2\varepsilon_{\mathbf{p}}}\tilde{\Delta}^{>}(1)\right]\\
-i(1-n_{\mathbf{p}})&\left[u_{\mathbf{p}}^2\Sigma^{<}(1) + v_{\mathbf{p}}^2\tilde{\Sigma}^{<}(1) + \frac{\Delta_{0,\mathbf{p}}^*}{2\varepsilon_{\mathbf{p}}}\Delta^{<}(1) + \frac{\Delta_{0,\mathbf{p}}}{2\varepsilon_{\mathbf{p}}}\tilde{\Delta}^{<}(1)\right],
\end{aligned} \tag{S23}$$

where $(1) = (\mathbf{p}, \omega = \tilde{\varepsilon}_{\mathbf{p}})$. Computing self-energies (Fig.S1) with condition $N_{\mathbf{p}} \to 0$ and holding skew scattering processes only we obtain expressions (21)-(23) of the main text.

5. The function F is expressed through a triple integral:

$$F(\gamma) = \gamma^7\int\limits_1^\infty dx\int\limits_1^\infty dy\int\limits_1^\infty dz\,\frac{xz(f_1+f_2+f_3+f_4+f_5+f_6)}{\sqrt{y^2-1}(x^2-1/4)\cosh^2(x\gamma/2)(z^2-1/4)}, \tag{S24}$$

where

$$
\begin{aligned}
f_1(\gamma) &= \Big[(1-n_y)\theta(x-y) - n_y\theta(y-x)\Big]\theta(y-z)\theta(z-x)(z-x)^3(y-z)^2,\\
f_2(\gamma) &= n_y\Big[\theta(y-z)(z+x)^3(y-z)^2 - \theta(x-z)(z+y)^2(x-z)^3\Big],\\
f_3(\gamma) &= (x-y)^2\Big[(1-n_y)\theta(x-y) - n_y\theta(y-x)\Big]\\
&\quad\times\Big\{\Big[\theta(x-z)(1-n_z) - \theta(z-x)n_z\Big](x-z)^3 + \Big[\theta(y-z)(1-n_z) - \theta(z-y)n_z\Big](y-z)^2(2y-x-z)\Big\},\\
f_4(\gamma) &= -(x+y)^2 n_y\\
&\quad\times\Big\{\Big[\theta(x-z)(1-n_z) - \theta(z-x)n_z\Big](x-z)^3 - \Big[\theta(y-z)(1-n_z) - \theta(z-y)n_z\Big](y-z)^2(2y+x-z)\Big\},\\
f_5(\gamma) &= (x-y)^2\Big[(1-n_y)\theta(x-y) - n_y\theta(y-x)\Big]n_z\Big[(x+z)^3 + (y+z)^2(2y-x+z)\Big],\\
f_6(\gamma) &= (x+y)^2 n_y n_z\Big[(x+z)^3 - (y+z)^2(2y+x+z)\Big],
\end{aligned}
\tag{S25}
$$

and

$$
n_y = \frac{1}{e^{y\gamma}+1}, \quad n_z = \frac{1}{e^{z\gamma}+1}. \tag{S26}
$$