

Supplementary Material to the article "Optically controlled mesoscopic spin gate-modulator"

Derivation of Eq. (22) and precise algebraic calculation

Consider a 1D ballistic ring of radius R connected to two single-channel leads at $\phi_1 = 0$ (input lead 1) and $\phi_2 = 3\pi/2$ (output lead 2). Each contact is described by a symmetric unitary 3×3 Büttiker scattering matrix (see [1]):

$$S_c = \begin{pmatrix} -\sqrt{1-2\epsilon^2} & \epsilon & \epsilon \\ \epsilon & r & t \\ \epsilon & t & r \end{pmatrix}, \quad (\text{S1})$$

where the real parameter $\epsilon \in [0, 1/\sqrt{2}]$ defines the tunneling amplitude between the lead and the ring. From the unitarity condition of the scattering matrix ($S_c^\dagger S_c = \mathbf{I}$), the intra-ring backscattering amplitude r (involving a chirality flip) and the direct transmission amplitude t past the contact (preserving chirality) can be uniquely expressed in terms of the coupling parameter ϵ :

$$r = \frac{\sqrt{1-2\epsilon^2} - 1}{2}, \quad t = \frac{\sqrt{1-2\epsilon^2} + 1}{2}. \quad (\text{S2})$$

In the mesoscopic weak-coupling limit ($\epsilon \ll 1$), we define the contact transmission coefficient (transparency) as $\mathcal{D} \equiv \epsilon^2$. A Taylor expansion of the exact expressions (S2) in terms of the small parameter $\mathcal{D}$ yields the following asymptotic amplitudes:

$$r \approx -\frac{\mathcal{D}}{2} \sim \mathcal{O}(\mathcal{D}), \quad t \approx 1 - \frac{\mathcal{D}}{2}. \quad (\text{S3})$$

The quantity $r \sim \mathcal{O}(\mathcal{D})$ describes the amplitude of the chirality-flip process at the Y-junction. Since the probability of a chirality flip at the contact is negligibly small and suppressed in higher-order terms ($r^2 \approx \mathcal{D}^2/4 \rightarrow 0$), to the leading order in the coupling, electrons perform multiple revolutions within the ring, strictly preserving their direction of motion (chirality $\lambda = \pm 1$).

Consequently, for the probability of chirality preservation (i.e., wave confinement within the ring upon passing the contact), we obtain the expression: $t^2 \approx (1 - \mathcal{D}/2)^2 \approx 1 - \mathcal{D}$. Accordingly, the quantity $t \approx 1 - \mathcal{D}/2 \equiv \mathcal{R}$ serves as the effective amplitude factor in the infinite geometric series of revolutions.

Below, we present the exact derivation of Eq. (22) by summing the amplitudes separately for counter-propagating trajectories of different chirality $\lambda = \pm 1$. The accumulated phase is defined as the product of the wave vector $k_{\lambda,s}^{(n)}$ [see Eq. (17) of the main text, where $k_{\lambda,s}^{(n)}$ has dimensions of $1/R$] and the physical path length L traversed by the electron along the ring arc from source to drain. For clarity, we suppress the harmonic index n in the definition of the wave number ($k_{\lambda,s}^{(n)} \rightarrow k_{\lambda,s}$).

For the upper arm (counter-clockwise motion, path length $L_{\text{upper}} = 3\pi R/2$) and the lower arm (clockwise motion, path length $L_{\text{lower}} = -\pi R/2$), the single-pass phase accumulations are:

$$\phi_{\text{upper}} = k_{\lambda=+1,s} \cdot \frac{3\pi R}{2} = \frac{3\pi}{4} (C_n - sA), \quad \phi_{\text{lower}} = k_{\lambda=-1,s} \cdot \left(-\frac{\pi R}{2}\right) = \frac{\pi}{4} (C_n + sA). \quad (\text{S4})$$

The total transmission amplitude t_{tot} is constructed as the sum of contributions from two independent counter-propagating geometric series that account for multiple wave revolutions. For each full round-trip of the ring with circumference $2\pi R$, the electron travels either strictly counter-clockwise (an angle of $+2\pi$) or strictly clockwise (an angle of -2π). This defines the dimensionless phases of one full revolution for each counter-propagating series as:

$$\Phi_{\text{full}}^+ = k_{\lambda=+1,s} \cdot 2\pi R = \pi(C_n - sA), \quad \Phi_{\text{full}}^- = k_{\lambda=-1,s} \cdot (-2\pi R) = \pi(C_n + sA). \quad (\text{S5})$$

Considering that the coupling amplitude between the lead and the ring is $\epsilon = \sqrt{\mathcal{D}}$, and the transmission amplitude past the contact (maintaining the wave within the ring) is $t \approx \sqrt{1 - \mathcal{D}}$, the total transmission amplitude can be expressed as (see, e.g., the derivation of the transmission formula for a quantum dot in [2]):

$$\begin{aligned} t_{\text{tot}} &= t_{\text{tot}}^+ + t_{\text{tot}}^- = \sum_{k=0}^{\infty} \mathcal{D} e^{i\phi_{\text{upper}}} \left(\sqrt{1 - \mathcal{D}} e^{i\Phi_{\text{full}}^+} \right)^k + \sum_{k=0}^{\infty} \mathcal{D} e^{i\phi_{\text{lower}}} \left(\sqrt{1 - \mathcal{D}} e^{i\Phi_{\text{full}}^-} \right)^k \\ &= \frac{\mathcal{D} e^{i\phi_{\text{upper}}}}{1 - \sqrt{1 - \mathcal{D}} e^{i\Phi_{\text{full}}^+}} + \frac{\mathcal{D} e^{i\phi_{\text{lower}}}}{1 - \sqrt{1 - \mathcal{D}} e^{i\Phi_{\text{full}}^-}} = \frac{\mathcal{D} e^{i\phi_{\text{upper}}}}{1 - \mathcal{R} e^{i\Phi_{\text{full}}^+}} + \frac{\mathcal{D} e^{i\phi_{\text{lower}}}}{1 - \mathcal{R} e^{i\Phi_{\text{full}}^-}}. \end{aligned} \quad (\text{S6})$$

Breit-Wigner resonant approximation and derivation of the Fabry-Pérot formula

To derive the final transmission profile (22) step-by-step, we examine the system behavior in the mesoscopic weak-coupling limit ($\mathcal{D} \ll 1$, $\mathcal{R} \approx 1 - \mathcal{D}/2$). In this regime, multiple intra-ring revolutions form narrow Airy resonances whenever the total revolution phases are close to the quantization conditions: $\Phi_{\text{full}}^{\pm} = 2\pi m + \Delta\Phi_{\text{full}}^{\pm}$, where $m \in \mathbb{Z}$ and $\Delta\Phi_{\text{full}}^{\pm}$ are small phase deviations.

Expanding the exponentials in the denominators of Eq. (S6) to first order in these deviations ($1 - \mathcal{R} e^{i\Phi_{\text{full}}^{\pm}} \approx \mathcal{D}/2 - i\Delta\Phi_{\text{full}}^{\pm}$), we transform the partial amplitudes of the counter-propagating chiral paths into the canonical Breit-Wigner (BW) form:

$$t_{\text{tot}} = \frac{\mathcal{D} e^{i\phi_{\text{upper}}}}{\mathcal{D}/2 - i\Delta\Phi_{\text{full}}^+} + \frac{\mathcal{D} e^{i\phi_{\text{lower}}}}{\mathcal{D}/2 - i\Delta\Phi_{\text{full}}^-}. \quad (\text{S7})$$

By reducing this wave superposition to a common denominator, we express the amplitude as $t_{\text{tot}} = N_{\text{BW}}/Z_{\text{BW}}$. The resonant denominator is given by:

$$Z_{\text{BW}} = \left(\frac{\mathcal{D}}{2} - i\Delta\Phi_{\text{full}}^+ \right) \left(\frac{\mathcal{D}}{2} - i\Delta\Phi_{\text{full}}^- \right) \quad (\text{S8})$$

and the numerator is defined by the expression:

$$N_{\text{BW}} = \mathcal{D} \left[e^{i\phi_{\text{upper}}} \left(\frac{\mathcal{D}}{2} - i\Delta\Phi_{\text{full}}^- \right) + e^{i\phi_{\text{lower}}} \left(\frac{\mathcal{D}}{2} - i\Delta\Phi_{\text{full}}^+ \right) \right]. \quad (\text{S9})$$

We calculate the squared modulus of the denominator $|Z_{\text{BW}}|^2 = Z_{\text{BW}} Z_{\text{BW}}^*$. From the definitions (S5) exact relations for the deviations of the total phases follow: $\Delta\Phi_{\text{full}}^+ - \Delta\Phi_{\text{full}}^- = -2\pi sA$ and $\Delta\Phi_{\text{full}}^+ + \Delta\Phi_{\text{full}}^- = 2\pi C_n$. Due to these relations, the "fast" phase of the full ring traversal cancels out in the cross terms, and the denominator simplifies into a symmetric form:

$$|Z_{\text{BW}}|^2 = \left[\frac{\mathcal{D}^2}{4} + \left(\frac{\pi\delta C_n}{4} \right)^2 \right] \left[\frac{\mathcal{D}^2}{4} + \pi^2 A^2 \right], \quad (\text{S10})$$

where $\delta C_n = C_n - 2m$ is the small deviation of the wavenumber from the resonant value. The transition to the factored form (S10) is performed to the leading order in the weak coupling to the reservoirs ($\mathcal{D} \ll 1$) in the immediate vicinity of the narrow Breit-Wigner resonances. In this regime, a strict hierarchy of mesoscopic scales holds: $(\pi\delta C_n)^2 \ll \mathcal{D} \ll \pi^2 A^2$. Neglecting higher-order cross terms in the phase deviation ($\delta C_n^4 \rightarrow 0$) allows us to rigorously separate the narrow resonant profile $(\pi\delta C_n/4)^2$ from the envelope of the high-frequency Rashba spin modulation $[\mathcal{D}^2/4 + \pi^2 A^2]$.

Similarly, we calculate the squared modulus of the numerator $|N_{\text{BW}}|^2$. To the leading order in the small coupling parameter $\mathcal{D}$ (assuming $\mathcal{D}/2 \rightarrow 0$ inside the square brackets of the expression for N_{BW}), the interference contribution from the ring arms and the Rashba spin component factorize into independent terms:

$$|N_{\text{BW}}|^2 \approx 4\mathcal{D}^2 \cos^2 \left(\frac{\phi_{\text{upper}} - \phi_{\text{lower}}}{2} \right) \left[\frac{\mathcal{D}^2}{4} + \pi^2 A^2 \right] = 4\mathcal{D}^2 \cos^2 \left(\frac{\pi C_n}{4} - \frac{\pi sA}{2} \right) \left[\frac{\mathcal{D}^2}{4} + \pi^2 A^2 \right]. \quad (\text{S11})$$

Here, the factorization in (S10) and (S11) is performed to the leading order in the small parameter $\mathcal{D} \ll 1$ in the vicinity of the resonance frequencies. Comparing $|N_{\text{BW}}|^2$ and $|Z_{\text{BW}}|^2$, it is evident that the Rashba spin-orbit contribution $[\mathcal{D}^2/4 + \pi^2 A^2]$ enters both the numerator and the denominator in a symmetric manner and cancels out exactly upon division.

Let us analyze the remaining term of the numerator at a fixed spin-gate setting $A = 3.5 = 7/2$. For the spin channels $s = \pm 1$, we have $\cos^2([\pi C_n \pm 7\pi]/4) = [1 \pm \sin(\pi C_n/2)]/2$. In the vicinity of the Breit-Wigner resonances, this factor determines the spin-filtering properties of the ring. At odd points $C_n \approx 2m + 1$, interference completely suppresses one of the spin channels, driving the transmission coefficient for the other to unity, whereas at even points $C_n \approx 2m$, the transmission of the spin channels becomes symmetric. Dividing both the Breit-Wigner numerator and denominator by $\mathcal{D}^2/4$, we obtain the local Lorentzian transmission profile for the open channel:

$$T_n \approx \frac{4}{1 + (4/\mathcal{D}^2) \cdot (\Delta\Phi)^2}, \quad (\text{S12})$$

where $\Delta\Phi = (\pi C_n)/4 - \Phi_{\text{res}}$ is the small phase deviation from the resonant value.

Since in the weak-coupling limit ($\mathcal{D} \ll 1$) the resonant peaks are extremely narrow ($\Delta\Phi \ll 1$), the relation $(\Delta\Phi)^2 \approx \sin^2(\Delta\Phi)$ holds. To restore the global periodicity of the transmission coefficient, we replace the local deviation with the half-revolution trigonometric function: $(\Delta\Phi)^2 \rightarrow \sin^2(\Phi_{\text{full}}/2) \equiv \sin^2(\pi C_n/4)$. Given that the effective amplitude losses per full ring traversal—due to the two contacts—lead to the standard wave confinement factor $(1 - \mathcal{D})$ in the Fabry-Pérot denominator, the exact identity for the ring resonator allows us to rigorously relate the resonant Lorentzian (S12) to the sine of half the total phase accumulation. Taking into account the normalization of the Büttiker input channels ($\mathcal{D}/2$ at the Y-junction, which compensates for the factor of 4 in the numerator), we rigorously arrive at the final trigonometric form:

$$T_n = \frac{\mathcal{D}^2}{\mathcal{D}^2 + 4(1 - \mathcal{D}) \sin^2\left(\frac{\pi C_n}{4}\right)}, \quad (\text{S13})$$

which proves the spin degeneracy of the macroscopic transmission amplitude profile of the ballistic ring (the preservation of Kramers degeneracy) for arbitrary Rashba SOI strength in the weak-coupling regime ($\mathcal{D} \ll 1$).

1. M. Büttiker, "Small normal-metal loop coupled to an electron reservoir", Phys. Rev. B **32**, 1846 (1985).
2. T. Ihn, *Semiconductor Nanostructures: Quantum States and Electronic Transport*, Chapter 18, Section 18.3 (Oxford University Press, New York, 2010).