Supplementary Material to the article "On the stability of the polar phase of superfluid ^{3}He in nematic aerogels"

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This Supplementary Material demonstrates that the spatial inhomogeneity of the local magnetization leads to a contribution to the scattering amplitude that is linear in the Berry curvature. A classical derivation of the Hamiltonian transformation for 2D electrons moving in a slowly varying magnetic field can be found in [1]. We consider the elastic scattering of a liquid ^{3}He quasiparticle by an impenetrable magnetic surface within the adiabatic approximation. The physical picture of the interaction is based on the separation of spatial scales: the barrier potential $V_0(z)$ varies on the scale of the Fermi wavelength ($\lambda_F \sim 1/k_F$), whereas the direction of the local magnetization $\mathbf{m}(\mathbf{r}_\parallel)$ varies smoothly on the macroscopic scale of the inhomogeneity or texture $R \gg \lambda_F$.

The total Hamiltonian of a quasiparticle moving in the free volume outside the aerogel strand is expressed via an effective spin-dependent surface potential of the layer:

$$\hat{\mathcal{H}}_{\text{total}} = \frac{\hat{\mathbf{p}}^2}{2M}\hat{\sigma}_0 + \hat{V}(\mathbf{r}_\parallel)\delta(z), \tag{S1}$$

where z is the coordinate along the local normal $\mathbf{n}$, and the matrix $\hat{V}(\mathbf{r}_\parallel)$ determines the local boundary scattering amplitude at the tangential point $\mathbf{r}_\parallel$:

$$\hat{V}(\mathbf{r}_\parallel) = -J\mathbf{m}(\mathbf{r}_\parallel)\boldsymbol{\sigma}, \tag{S2}$$

where $\mathbf{m}$ is the local magnetization vector. Applying a unitary transformation $\hat{U}(\mathbf{r}_\parallel)$ such that $\hat{U}(\mathbf{r}_\parallel)\mathbf{m}(\mathbf{r}_\parallel)\boldsymbol{\sigma}\hat{U}^\dagger(\mathbf{r}_\parallel) = \sigma_z$, the transformed Hamiltonian takes the form:

$$\hat{\mathcal{H}} = \frac{\hat{p}_\perp^2\sigma_0}{2M} + \frac{(\hat{\mathbf{p}}_\parallel\sigma_0 - \hat{\mathbf{A}}_\parallel)^2}{2M} - J\sigma_z\delta(z), \tag{S3}$$

where

$$\hat{A}_{\parallel,i}(\mathbf{r}_\parallel) = i\hbar\hat{U}^\dagger(\mathbf{r}_\parallel)\nabla_{\parallel,i}\hat{U}(\mathbf{r}_\parallel) = \hat{A}^D_{\parallel,i} + \hat{A}^{ND}_{\parallel,i}, \tag{S4}$$

where $\hat{A}^D_{\parallel,i} = A^{(z)}_{\parallel,i}\hat{\sigma}_z$ is the diagonal (adiabatic) part, and $\hat{A}^{ND}_{\parallel,i} = A^{(x)}_{\parallel,i}\hat{\sigma}_x + A^{(y)}_{\parallel,i}\hat{\sigma}_y$ is the off-diagonal part of the Berry potential, which describes texture-induced spin-flip transitions. The operator $\hat{A}$ is a column of 2×2 matrices. By definition, the Berry potential vanishes along the surface normal ($A_\perp = 0$). If the potential varies slowly, the off-diagonal part can be neglected; thus, to first order in the gradient, the local texture only causes a phase shift of the wave functions. To construct the effective scattering Hamiltonian, we expand the operator $\hat{A}^D_{\parallel,j}$ near the reflection point in terms of $\mathbf{r}_\parallel$:

$$A^D_{\parallel,j}(\mathbf{r}_\parallel) \approx A^D_{\parallel,j}(0) + \hat{r}_{\parallel,k}\left(\nabla_{\parallel,k}A^D_{\parallel,j}\right)_0. \tag{S5}$$

Choosing a gauge where $A^D_{\parallel,j}(0) = 0$, we consider the term linear in the tangential momentum $\hat{p}_{\parallel,j} = -i\hbar\nabla_{\parallel,j}$:

$$\hat{\mathcal{H}}_{int} = -\frac{1}{2M}\left(\hat{p}_{\parallel,j}\hat{A}^D_{\parallel,j} + \hat{A}^D_{\parallel,j}\hat{p}_{\parallel,j}\right). \tag{S6}$$

Substituting the expansion and simplifying yields a term proportional to the Berry curvature:

$$\hat{\mathcal{H}}^{(a)}_{int} = -\frac{1}{2M}\left(\nabla_l A^D_j - \nabla_j A^D_l\right)\hat{r}_l\hat{p}_j\cdot\sigma_z = e_{ljk}H_k\hat{r}_l\hat{p}_j\cdot\sigma_z, \tag{S7}$$

where the expression for the Berry curvature follows from the definition of the gauge potential:

$$H_z = \frac{\hbar}{2}\mathbf{m}\cdot\left[\frac{\partial\mathbf{m}}{\partial x}\times\frac{\partial\mathbf{m}}{\partial y}\right] = \frac{\hbar}{2}\sin\theta\left(\frac{\partial\theta}{\partial x}\frac{\partial\varphi}{\partial y} - \frac{\partial\theta}{\partial y}\frac{\partial\varphi}{\partial x}\right), \tag{S8}$$

the angles θ and ϕ parameterize the direction of the magnetization $\mathbf{m}$. Transforming back to the laboratory frame results in the effective spin-orbit interaction used in this work:

$$\mathcal{H}_{eff} \sim (\mathbf{m}\cdot\boldsymbol{\sigma})\cdot(\mathbf{H}\cdot[\mathbf{r}\times\mathbf{p}]). \tag{S9}$$

The adiabatic approximation allows for an explicit derivation of the antisymmetric tensor h_{ij} associated with the Berry curvature of the magnetization texture. Since its form is determined by the symmetry of the texture, non-adiabatic processes (off-diagonal terms) beyond the adiabatic approximation are expected to primarily renormalize the coefficient of this invariant without altering its tensor structure. Therefore, the description via the Berry curvature should be understood as a qualitative microscopic justification rather than a quantitative estimation of its magnitude. The quantitative relation between h_{ij} and the parameters of the magnetic texture requires a theory beyond the adiabatic limit and lies outside the scope of this work. For instance, accounting for

the off-diagonal terms in the gauge potential also contributes to the phase shift of the initial-state wave function; in the second order of gradient perturbation theory, this leads to an antisymmetric fourth-order tensor with respect to the texture gradients, etc. Thus, the small expansion parameter is the ratio $\frac{\hbar^2}{2ML^2J_0}$, where L is the scale of the magnetic inhomogeneity. For $L \sim 30$ nm and $J_0 \sim 1$ mK, this parameter is of the order of 10^{-1}.

The actual structure of the magnetic coatings on the aerogel strands is unknown. However, the presence of strand intersections, thickenings, and other inhomogeneities leads to spatial variations in the local magnetization direction both along and across the strands. Such textures possess a non-vanishing local chirality $\mathbf{m} \cdot (\nabla_i \mathbf{m} \times \nabla_j \mathbf{m})$, which within the adiabatic approximation gives the antisymmetric tensor h_{ij}.

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1. P. Bruno, V. K. Dugaev, M. Taillefumier, "Topological Hall Effect and Berry Phase in Magnetic Nanostructures", Phys. Rev. Lett., **93**, 096806 (2004).